DESIGN OF CONTROL CHARTS FOR THE MEAN VECTOR OF A MULTIVARIATE NORMAL PROCESS

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To my Mother and Father.

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SUMMARY

The purpose of the investigation reported in this thesis was to develop a generalized cost model capable of managing certain multiple quality characteristics. A general method for choosing the sample size, frequency of sampling and critical region for multiple characteristic quality control tests is presented. To accomplish this objective, a general expected cost model was developed by treating movement of the process parameter as a finite Markov Chain. The process parameter in question is the mean vector of the process. A general method for calculating the steady state probabilities of the process being in or out of control is presented. From these steady state probabilities the expected cost is calculated.

A two-stage numerical procedure is developed and programmed for a digital computer to determine the sample size (N), the interval between samples (K) and the critical region parameter (T_{α}^2) which minimizes the expected cost.

The two-stage procedure is used to determine the optimal values of N, K and T_{α}^2 for quality control tests of the mean vector of a bivariate normal process. The optimal test parameter values are tabulated for various values of the fixed cost of taking a sample and making the test, the cost of inspecting a unit, the cost of investigating and correcting a process and a number of example problems.

CHAPTER I

INTRODUCTION

This chapter is an overview of this investigation into multivariate quality control procedures, and is organized as follows: (a) statement of the problem to be investigated, and (b) survey of pertinent literature. Theories presented and solution methodology proposed preliminarily in this chapter will be more thoroughly discussed in corresponding explanatory chapters.

Control of Multiple Quality Characteristics

An efficient monotoring system to control the stability of a random variable process is dependent upon reliable statistical quality control procedures. Patterns of variations revealed by these procedures attest to the stability of the system being used. Those variations within specified limits are accepted. For those variations outside these specified limits, specific causes must be sought and justified in order that the variation outside the stable pattern may be corrected.

Many industrial processes are characterized by two or more quality characteristics. For example, in the production of axles the length, diameter, and weight may all be important quality characteristics. The use of univariate methods in an attempt to control a process characterized by two or more statistically independent quality characteristics would result in substantial error from the independent testing and control of each variable. Therefore, the simultaneous control of two or more related variables when the joint effect of these variables determines quality, is very important.

To illustrate this point, consider the following situation: Two quality characteristics x and y are distributed according to a bivariate normal distribution with equal variances. Independent testing and control of each variable implies the use of separate control charts for each variable. If the control limits are set at plus and minus three standard deviations on each variable then

$$\Pr(\bar{x}>\bar{\bar{x}}\pm 3\sigma_{\bar{x}}) = 0.003$$

and

$$\Pr(\bar{y}>\bar{\bar{y}}\pm 3\sigma_{\bar{y}}) = 0.003$$

Therefore, the process is considered to be in control if $\bar{x}$ and $\bar{y}$ do not exceed their long term averages by three standard deviations. However, note that the probability that *both* characteristics are simultaneously in control is given by

$$[1-\Pr(\bar{x}>\bar{\bar{x}}+3\sigma_{\bar{x}})][1-\Pr(\bar{y}>\bar{\bar{y}}+3\sigma_{\bar{y}})] = (0.997)(0.997) = 0.995$$

and the true probability of Type I error is 1 - 0.995 = 0.005. If the two characteristics are not statistically independent, then the true probability of Type I error may not be so easily determined. In any case, it is evident that the probability of Type I error is distorted badly by using univariate control procedures on multivariate data. Further, this example illustrates conditions in which the variance of both variables is equal and the critical region parameter is 3σ. It follows, then, that the errors described above are compounded and become more serious when the control limits are set at values less than 3σ, or when the quality characteristics are not statistically independent.

In a majority of instances, the variances are not equal and the quality characteristics are not statistically independent. Therefore, control procedures must be based on multivariate statistical techniques which results in the control region becoming an ellipse. This correctly determines the error probabilities involved.

These errors manifest themselves by incorrectly determining the state of the process which may be considered either in or out of control. The quality characteristics characterizing a process are considered random variables and are often highly interrelated. Therefore, errors and increased costs result from the independent application of univariate control methods to each characteristic.

Problems of increasing errors and their associated costs have intensified in recent years and adequate techniques for multivariate quality control are necessary. Adequate detection of changes in the process parameters is mandatory, and achievement of this detection is contingent upon reliable process control techniques.

Purpose and Scope of the Thesis

Advances in multivariate statistical quality control procedures have, until recently, primarily concerned theoretical matters. Few suggestions were offered to advance the application of these multivariate procedures to practical problems. However, considerable attention has been directed toward improving techniques involving only a single quality characteristic. Several investigations were devoted to determining the various costs involved in univariate control tests. These endeavors have produced expected cost models which effectively ascertain optimal values for certain test parameters. The determination of optimal parameters result in minimum costs of the quality control procedures, regardless of the state of the process.

The usual test parameters explicated in various univariate cost models in the literature include sample size, the time interval between samples, and the statistic defining the critical region. The effectiveness of these cost models is compromised, however, by two serious limitations. First, the quality characteristic used to control the process must be univariate in nature. Secondly, these models require an unrealistic assumption, namely that the population variance is known.

Further study of this general problem area is indicated, because many products have two or more quality characteristics which vary continuously, and exercise great influence on the ultimate function of the product. Furthermore, if the population variance is assumed, or must be assigned subjective values, the procedures involved in the cost models of other authors cannot be effectively applied in many realistic quality control situations.

A typical observation which corroborates the effect of these limitations is offered by Morrison. He indicates that the majority of practical situations require estimation of the population variance (in the univariate case) or covariance matrix (in the multivariate case) from a sample when the test of hypothesis involves the mean or mean vectors. Considering that controlling n characteristics would require knowledge of n variances, plus n(n-1) covariances, Morrison's observation is quite important.

The goal of this investigation is the development of a generalized cost model capable of managing certain multiple quality characteristics. It is assumed that these quality characteristics are continuous random variables. Furthermore, this model should not require prior knowledge of the population covariance structure. If successful, this model would determine certain vital information for the multivariate quality control case: the optimal sample size, the optimal sampling interval, and the optimal critical region parameters. The application of this information would, hopefully, yield procedures for minimizing total costs of operating multivariate quality control systems.

The scope of this inquiry will be restricted to an investigation of quality control test procedures for maintaining control of the means of the quality control characteristics. Further, the quality characteristics are assumed to be described by a p-dimensional random variable following a multivariate normal distribution with mean vector, $\underline{u}$, and variance-covariance matrix $\underline{V}$. Furthermore, knowledge of the probability of occurrence of an assignable course, and of various costs is also assumed to be known.

Survey of the Literature

The importance of multivariate statistical procedures was recognized by Hotelling (13) as early as 1931. Generalization of the student's "t" distribution provided Hotelling with a basis for development of a unique statistical procedure. This procedure broadened the horizons of multivariate analysis. It could be utilized in situations involving more than one variable, and in addition, those instances in which the variance must be estimated from a sample. Hotelling (13) showed that if N observations from a multivariate normal population with mean vector $\underline{u}$ and variance-covariance matrix $\underline{V}$ have been recorded, and the variance-covariance matrix $\underline{V}$ is estimated from the sample, by the statistic $\underline{S}$, then the quadratic form

$$T^2 = N(\bar{\underline{x}}-\underline{u}_0)'\underline{S}^{-1}(\bar{\underline{x}}-\underline{u}_0)$$

is distributed $T^2_{p,N-1}$ with frequency function

$$f(T^2) = \frac{2\Gamma(N/2)T^{p-1}}{\Gamma(p/2)\Gamma[(N-p)/2](N-1)[1+T^2/(N-1)]^{N/2}}$$

Hotelling's 1931 offering has more recently been extended by Hicks (12), Jackson (15,16), and Kramer, Clyde and Jensen (18,19,20,21). These authors have unanimously endorsed the importance of Hotelling's techniques, and have demonstrated the utility of these techniques in practical situations.

A publication by Ghare and Torgersen (9) provides an interesting example of the type of error that may frequently occur when dealing with the multivariate case and unknown variance. The authors attempted to examine the procedure involved in controlling two or more variables simultaneously. They supported their discussion with an example of a bivariate control chart. This example uses the variance-covariance matrix which had been estimated from a random sample. Their procedure utilized a χ^2 test which, in this case, was incorrect as the χ^2 statistic is applicable only when the variance-covariance matrix is known and not estimated from sample data. The chi-square test statistic is given by

$$\chi^2 = N(\underline{\bar{x}}-\underline{u}_0)'\underline{V}^{-1}(\underline{\bar{x}}-\underline{u}_0),$$

where the frequency function of the chi-square distribution is

$$f(\chi^2) = \frac{(\chi^2)^{(p-2)/2}\exp[-\chi^2/2]}{2^{p/2}\Gamma(p/2)}$$

The appropriate test should be based on the Hotelling's T^2 distribution which has been discussed in the previous paragraph.

The multivariate quality control literature does not contain a general method for determining the optimal test parameters. The only existing method suggested for this purpose is discussed by Morrison (22) and Anderson (2). This method is based on selection of a sample size and a critical region by arbitrary specification of the probability of the Type I error (q_0) and the probability of Type II error (q_1). This, essentially, specifies the magnitude of changes in the mean vector components that it is desirable to detect with probability q_1 for a given value of q_0.

The flaw in this approach is the lack of consideration of the frequency of sampling. Furthermore, the arbitrary choice of Type I and Type II error probabilities is no more defensible than an arbitrary choice of the critical region and the sample size.

Nothing has been reported in the literature concerning the costs involved in controlling a multivariate process. Several authors have developed cost models for univariate statistical control procedures. Therefore, as a precedent to development of a multivariate cost model, those cost models involving a single process parameter will be summarized. Those models chosen for review will be ones in which the assumptions made concerning the process are comparably simplified. Cowden (4) developed a cost model to study the economic design of tests for the mean of a process. The form of his model is

$$C = C_0 + C_1 + C_2$$

In this model the three costs involved are the operating cost, the engineering cost and the merchandising cost. The operating cost is defined as that associated with the testing procedure. The engineering cost involves the costs of investigating a process when there has been an apparent shift in the mean. The merchandising cost is associated with costs incurred when the process is not in control and defective items are produced.

To develop these costs Cowden makes the following assumptions:

1. The process is considered to be out of control at the beginning of each day.
2. Once the trouble is detected, it is corrected quickly and no further trouble can occur that day.
3. The cost of looking for trouble is proportional to the shift in mean.
4. The probability of finding trouble is a function of the cost of looking for trouble.

The value of this model is minimized by unrealistic assumptions, particularly those assumptions concerning the status of the process at the beginning of each day, and the assumption that trouble occurs only at the beginning of the day. Furthermore, the third assumption involves the formulation of an equation for determining the cost of looking for trouble for a given shift in the mean. Since these costs are rarely available in actual practice, any equation used to obtain them must be developed in an arbitrary manner.

The model of Duncan (5) involves maximizing the profit from a process instead of minimizing the cost of controlling quality. His model is of the form:

$$\text{Profit} = \text{Income} - \text{Cost}$$

Since Duncan assumes that income is independent of the quality control test chosen, maximizing profit is equivalent to maximizing cost. Therefore, the cost model would be:

$$C = C_0 + C_1 + C_2$$

In this model, C_0 is the average cost per hour for operating the quality control procedure, C, is the average cost per hour of looking for trouble, and C_2 is the average cost per hour of producing defectives.

The following assumptions are made in the development of Duncan's model: When a shift in the mean occurs, it shifts by a constant amount. Secondly, that the average time the process will remain in control is $1/\lambda$ hours and the probability of shifting out of control between the time t and $t + \Delta t$ is approximately $\lambda e^{-\lambda t}\Delta t$ for small Δt. His method of handling the probability of the process mean shifting to an out-of-control state seems to be very realistic. However, the equation for determining the costs involved in looking for trouble and producing defectives seems to be simplified with respect to the probability involved.

Knappenberger (17) developed the most comprehensive cost model for univariate quality control. The total cost per unit of product associated with the testing procedure is

$$C = C_0 + C_1 + C_2$$

where C_0 is the cost associated with the sampling and testing procedure, C_1 is the cost associated with rejecting the null hypothesis of statistical control, and C_2 is the cost associated with the production of defective units. Since C_0, C_1, or C_2 may be random variables, the general model is the expected value of the three costs involved, or

$$E(C) = E(C_0) + E(C_1) + E(C_2)$$

In this model it is assumed that the process may have more than two existing states, and that the process parameter is a continuous random variable. It is assumed that this parameter can be approximated by a discrete random variable.

The cost of investigating and correcting a process which is out of control is also assumed to be a random variable. When the process goes out of control, the model assumes that it remains in that state until detected. However, the process can shift farther out of control before being detected.

The complex methods required for determining the various probabilities involved is the only limitation of Knappenberger's work. These complexities result when the mean is allowed to shift to more than one out-of-control state. The optimal values of the test parameters provide another interesting aspect with regard to the complexities of the model. The optimal test parameters obtained when only one out-of-control state was allowed are not significantly different from those obtained when 6 or 12 similar states are considered. This occurrence may be attributed to the fact that regardless of the number of possible out-of-control states of the process, there is no effect on the optimal critical region parameter or the optimal time between samples. In addition, the effect on the optimal sample size does not seem large enough to warrant the increased calculations required when allowances are made for the process to shift to a number of out-of-control states.

Taylor (25) has published recent developments in optimal control procedures for the univariate case. Taylor's theory is based on the assumption that the process is either in or out of control. The important results of this work established that a fixed sample size and sampling interval will yield nonoptimal results. A more effective procedure stated that sampling should be determined at each stage by current posterior probabilities. It appears that these suggestions have theoretical value, but minimal relevance to industrial application due to the continuous modification which would be required.

The obviously increased complexities involved in controlling a multivariate process requires that the cost model developed for this purpose be kept as simple as possible. In doing so, hopefully, a useful tool for industrial application will result.

This investigation will seek to develop such a model, operating under the assumption that the process parameter is either in or out of control. However, a number of different values of the magnitudes of the shift will be considered. Although non-optimal results have been previously obtained with a fixed sample size and sampling interval, the cost model developed will consider these values fixed. Fixed values are being applied in this case since it would be impossible to develop a general model capable of managing any number of quality characteristics if these values were not fixed. The same case applies to a situation in which the process is allowed to be out of control more than once. More specifically, relaxing either restriction stated above would demand development of a different cost model each time the number of quality characteristics changed. This alternative is obviously unacceptable; therefore, the ensuing model is developed with the limits of aforementioned practical restrictions.

CHAPTER II

DEVELOPMENT OF THE MATHEMATICAL MODEL

Quality Control Charts monitor the stability of a random variable and can be used to maintain control of a production process. This utility has established control charts as one of the most important monitoring devices for industrial management. A principal objective of management is control of the quality characteristics generated by their processes. It follows that the greater the degree of control obtained, the more efficient and reliable the resultant procedure will be. There are two advantages of effective and efficient quality control. First, a reduction in the cost of inspection and the cost of rejection. Second, the attainment of maximum benefits from quality production due to more uniform quality of the products. Realization of these advantages has required the development of an effective method for determining the costs involved in the quality control procedure. This chapter will develop a cost model appropriate for the multivariate situation.

General Assumptions and Nomenclature

A general, minimum expected cost procedure is developed to determine optimum values for the following parameters for a process whose output is described by multiple quality characteristics: sample size, frequency of sampling, and the critical region for testing hypotheses for the multiple quality characteristics. The following notation will be employed:

N is the sample size in units of product.

k is the total number of units of product produced between samples, which includes the units sampled.

$\underline{x}$ is the quality characteristic, such that $\underline{x} \sim N(\underline{u},\underline{V})$.

$\underline{u}$ is the mean vector of the quality characteristics being tested.

$\underline{V}$ is the variance-covariance matrix of $\underline{x}$.

$\bar{\underline{x}}$ is the vector of sample averages and is assumed to have a multivariate normal distribution with mean vector $\underline{u}$ and variance-covariance matrix $\underline{V}/N$.

T^2 is the test statistic, such that $T^2 \sim T^2_{p,N-1}$. Reject H_0 if $T^2 > T^2_{\alpha,p,N-1}$. $T^2 = N(\bar{\underline{x}}-\underline{u}_0)\underline{S}^{-1}(\bar{\underline{x}}-\underline{u}_0)$.

$\underline{S}$ is an unbiased estimator of $\underline{V}$ based on a sample of size N.

p is the number of quality characteristics.

$\underline{u}_0$ is the value of the mean vector when the null hypothesis is true. (When all components of $\underline{u}$ equal $\underline{u}_0$.)

$\underline{u}_1$ is the value of the mean vector when the alternative hypothesis is true. (When one or more components of the mean vector is out of control.)

$\underline{q}$ is a vector of probabilities (q_0,q_1), where q_0 and q_1 are the probabilities of rejecting H_0 when $\underline{u} = \underline{u}_0$ and $\underline{u} = \underline{u}_1$, respectively.

$\underline{\alpha}$ is a vector of probabilities (α_0,α_1), where α_0 and α_1 are the steady state probabilities that $\underline{u} = \underline{u}_0$ and $\underline{u} = \underline{u}_1$, respectively, at the time the test is made.

$\underline{f}$ is a vector of probabilities (f_0,f_1) where the probability of producing a defective item is f_0 and f_1 when $\underline{u} = \underline{u}_0$ and $\underline{u} = \underline{u}_1$, respectively.

$\underline{\gamma}$ is a vector of probabilities (γ_0,γ_1), where the probabilities of $\underline{u} = \underline{u}_0$ and $\underline{u} = \underline{u}_1$ at any point in time are γ_0 and γ_1, respectively.

The quality control procedure used to maintain control of a process involves testing the null hypothesis

$$H_0: \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_p \end{bmatrix} = \begin{bmatrix} u_{01} \\ \cdot \\ \cdot \\ \cdot \\ u_{0p} \end{bmatrix}$$

against the alternative hypothesis

$$H_1: \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_p \end{bmatrix} \neq \begin{bmatrix} u_{01} \\ \cdot \\ \cdot \\ \cdot \\ u_{0p} \end{bmatrix}$$

The testing procedure is accomplished by randomly selecting N independent observation vectors, each containing p responses. These observation vectors may be represented by

$$\underline{x}_i = \begin{bmatrix} x_{i1} \\ \cdot \\ \cdot \\ \cdot \\ x_{ip} \end{bmatrix}, \quad i=1,\ldots,N.$$

From these N vectors the vector of sample averages $\bar{\underline{x}}$ is calculated as

$$\bar{\underline{x}} = \frac{1}{N} \sum_{i=1}^{N} \underline{x}_i \tag{2.1}$$

The central limit theorem indicates that as N becomes large the limiting distribution of $\underline{\bar{x}}$ can be assumed to approach a multivariate normal distribution. The quadratic form T^2 of this distribution is given by

$$T^2 = N(\underline{\bar{x}}-\underline{u}_0)'\underline{S}^{-1}(\underline{\bar{x}}-\underline{u}_0) \tag{2.2}$$

where $\underline{S}$ is the estimate of $\underline{V}$, the variance-covariance matrix of the vector $\underline{\bar{x}}$. Hotelling (13) showed that the quadratic form T^2 has a $T^2_{p,N-1}$ distribution.

To construct a control chart and define the confidence region, tables of the central F distribution can be used since

$$T^2_{p,N-1} = \frac{p(N-1)}{N-P} F_{p,N-p},$$

when the null hypothesis is true. Therefore, accept H_0 if

$$T^2 \leq T^2_{\alpha,p,N-1} = \frac{p(N-1)}{N-p} F_{\alpha,p,N-p}$$

and reject H_0 if

$$T^2 > T^2_{\alpha,p,N-1}$$

Graphically this amounts to accepting H_0 unless a value of T^2 falls above the upper control chart limit (UCL) defined by $T^2_{\alpha,p,N-1}$. Figure 2.1 illustrates such a control chart.

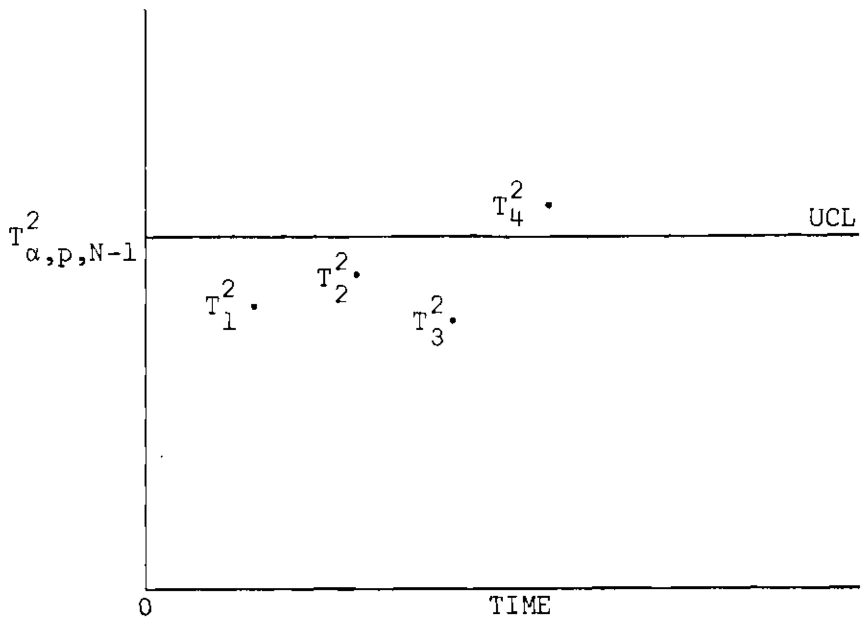

Figure 2.1 T^2 Control Chart

In Figure 2.1 the value T_4^2 would result in rejection of the null hypothesis, whereas the other values shown would not. Obviously, larger values of T^2 indicate the process is further from the desired standards. Therefore T^2 charts require only an upper limit. In addition, the control chart makes use of one overall measure obtained from each combination of sample values. The advantages of using such a chart is that the status of the process can be characterized by one number and the chart retains the time scale, thereby summarizing the condition of the process over a period of time. The only disadvantage of the chart is that when an out-of-control condition exists, it is necessary to consider the original data to determine which component of the mean vector is out of control. Procedures to perform this task are not well defined.

General Form of the Model

The total cost (C) of the testing procedure for each unit of product involves the following three component costs:

$$C = C_1 + C_2 + C_3$$

where

C_1 is the cost of testing and sampling.

C_2 is the cost involved in rejecting the null hypothesis, H_0.

C_3 is the cost associated with producing defective units when H_0 is not rejected.

The expected value of the total cost equation will be utilized since the various costs involved may be random variables. That is,

$$E(C) = E(C_1) + E(C_2) + E(C_3) \tag{2.3}$$

Expected Cost of Sampling and Testing

Cowden (4), Duncan (5), and Knappenberger (17), assumed that the cost of sampling and testing consisted of a constant amount, independent of the number of units sampled, plus a constant amount for each unit sampled. In view of the difficulty of obtaining accurate cost estimates, more complex cost functions appear to be unwarranted. Thus,

$$C_1 = a_1 + a_2 N,$$

where

a_1 is the fixed cost per sample.

a_2 is the cost per unit of product sampled.

Dividing the above cost by the number of units produced between samples, k, the sampling and testing cost per unit is

$$C_1 = \frac{a_1}{k} + \frac{a_2 N}{k} .$$

For any given sampling plan, each of the components of the above cost is a constant; hence, the expected sampling and testing cost per unit is just

$$E(C_1) = \frac{a_1}{k} + \frac{a_2 N}{k} . \tag{2.4}$$

Expected Cost of Rejecting H_0

Knappenberger (17) has considered the cost of rejecting the null hypothesis as the cost of determining and correcting the cause of an apparent shift of the process parameter $\underline{u}$ from $\underline{u}_0$, to some new value. Therefore, the cost per unit of rejecting the null hypothesis in a given sampling period is

$$C_2 = \frac{yz}{k}$$

Here y is a random variable which assumes the value of one if the null hypothesis is rejected, and the value of zero if the null hypothesis is not rejected. z represents the cost of investigating and correcting the process.

The true value of the parameter $\underline{u}$ is necessary to determine the cost of correcting a given shift in the process parameter. The cause of small shifts will be more difficult to find than the cause of large shifts. Yet, the cost of correcting the process after the cause has been determined can be assumed to be correspondingly larger for large shifts than for small shifts.

It is improbable that prior information would be available regarding cost in a situation where the cost of correcting a process is a function of the true value of $\underline{u}$. If prior information is available, it would probably concern the number of times the process goes out of control, the length of time the process is inoperable, and the cost per hour of an inoperable process. From this information, the average cost of getting the process back into operation can be determined with reasonable accuracy.

It will be assumed, then, that the cost of investigating and correcting a process apparently out of control is a random variable, say z, whose distribution does not depend on the parameter $\underline{u}$. It is further assumed that the random variables y and z are stochastically independent. Therefore, the expected cost per unit of rejecting H_0 is

$$E(C_2) = \frac{E(y)E(z)}{k}$$

The expected value of y is the probability that the null hypothesis is rejected (i.e., the probability that y equals one). The expected value of z is the average cost of investigating and correcting the process.

If the expected value of $z = a_3$, then the expected cost per unit associated with rejecting H_0 is

$$E(C_2) = \frac{a_3 Pr(y=1)}{k}$$

The unconditional probability of rejecting H_0 is equal to the sum of the conditional probability of rejecting H_0 given that the process parameter is in control and the conditional probability of rejecting H_0 given that the process parameter is out of control at the time the test is performed. Therefore,

$$Pr(y=1) = q_0\alpha_0 + q_1\alpha_1$$

Hence, the expected cost is

$$E(C_2) = \frac{a_3}{k}[q_0\alpha_0+q_1\alpha_1] \tag{2.5}$$

Expected Cost of Accepting H_0

The costs involving production of defective items is the major penalty associated with incorrectly accepting H_0. There is evidence to indicate that the relationship between the number of defectives produced and the cost of producing defectives is non-linear. This assertion is reasonable because a small number of defectives may go unnoticed by the consumer, while a large number of defectives may cause loss of future business. The nature of this non-linear relationship would be difficult, if not impossible to determine; therefore, a more simple linear relationship will be assumed.

If a_4 represents the cost associated with producing a defective unit of product, the cost per unit associated with accepting H_0 is

$$C_3 = a_4 w$$

In this case, w is a random variable which takes on the value of one if the unit is defective, and the value of zero if the unit is not defective. Thus, the expected cost of accepting H_0 is

$$E(C_3) = a_4 Pr(w=1)$$

The unconditional probability that a unit is defective is the sum of the conditional probabilities of producing a defective when the mean is in control (out of control) times the probability that the process is in control (out of control) at any time. Thus, the expected cost of accepting H_0 is given by

$$E(C_3) = a_4[f_0\gamma_0+f_1\gamma_1] \tag{2.6}$$

Development of Probability Vectors

This section will develop explicit definitions for the probability vectors $\underline{q}$, $\underline{\alpha}$ and $\underline{\gamma}$. Obviously, these play an important role in model construction.

The probability of rejecting H_0 is defined as the probability vector $\underline{q} = (q_0, q_1)$ where q_0 is the probability of rejecting H_0 when the process is in control and q_1 is the probability of rejecting H_0 when the process is out of control. Previously, it was assumed that the vector of sample means $\underline{\bar{x}}$ had a multivariate normal distribution, with mean $\underline{u}$ and variance-covariance matrix $\underline{V}/N$. Thus, the probability of rejecting H_0 when the process is in control is

$$q_0 = Pr(T^2 > T^2_{\alpha,p,N-1}),$$

or

$$q_0 = \int_{T^2_{\alpha,p,N-1}}^{\infty} f(u)du \tag{2.7}$$

In this case f(u) is the central T^2 distribution with p and N-1 degrees of freedom and is given by

$$f(u) = \frac{2\Gamma(N/2)u^{p-1}}{\Gamma(p/2)\Gamma[(N-p)/2](N-1)^{p/2}[1+u^2/(N-1)]^{N/2}}$$

Obviously, q_0 is just the probability of Type I error. The probability of rejecting H_0 when the process is out of control is

$$q_1 = \int_{T^2_{\alpha,p,N-1}}^{\infty} f(u)du, \tag{2.8}$$

where $f(u)$ is the noncentral T^2 distribution with degrees of freedom p and N-1 and noncentrality parameter λ. The frequency function of the noncentral T^2 distribution is given by

$$f(u) = \sum_{i=0}^{\infty} \frac{2\Gamma[(2i+N)/2]\lambda^{i}u^{2i+p-1}}{\Gamma[(2i+p)/2]\Gamma[(N-p)/2]N^{[(2i+p)/2]}i![1+T^2/(N-1)]^{[(2i+N)/2]}}$$

The probability q_1 is generally referred to as the power of the test. The values for q_0 and q_1 in the above expression can be obtained from tables of the central F and noncentral F distributions. When the null hypothesis is true, the quantity

$$F = \frac{N-p}{p(N-1)} T^2$$

has the central F distribution with degrees of freedom p and N-p. Therefore, equation 2.7 can be replaced with

$$q_0 = \int_{\frac{N-p}{p(N-1)} T^2_{\alpha,p,N-1}}^{\infty} f(F,p,N-p)dF, \tag{2.9}$$

where $f(F,p,N-p)$ is the central F distribution with degrees of freedom p and N-p.

When the alternative hypothesis is true, the quantity

$$F' = \frac{N-p}{p(N-1)} T^{2'}$$

has the noncentral F distribution with degrees of freedom p and N-p and noncentrality parameter λ. Thus, equation 2.8 can be replaced with

$$q_1 = \int_{\frac{N-p}{p(N-1)} T^2_{\alpha,p,N-1}}^{\infty} f(F',p,N-p,\lambda)dF', \tag{2.10}$$

where $f(F',p,N-p,\lambda)$ is the noncentral F distribution with p and N-p degrees of freedom and noncentrality parameter λ. The frequency function $f(F,p,N-p)$ is given by

$$f(F,p,N-p) = \frac{\Gamma(N/2)[p/(N-p)]^{p/2}F^{(p-2)/2}}{\Gamma(p/2)\Gamma[(N-p)/2][1+pF/(N-p)]^{N/2}}$$

For the noncentral F distribution the frequency function is

$$f(F',p,N-p,\lambda) = \sum_{i=0}^{\infty} \frac{\Gamma[(2i+N)/2][p/(N-p)]^{[(2i+p)/2]}\lambda^i e^{-\lambda} F^{[(2i+p-2)/2]}}{\Gamma[(N-p)/2]\Gamma[(2i+p)/2]i![1+pF/(N-p)^{[(2i+N)/2]}}$$

The next probability vector to be considered will be $\underline{\alpha} = (\alpha_0,\alpha_1)$, where α_0 is the steady state probability that the process mean is in control at testing time, and α_1 is the steady state probability that the process mean is out of control when the test is conducted. To develop the probability vector $\underline{\alpha} = (\alpha_0,\alpha_1)$, a transition probability matrix ($\underline{B}$) is required. The elements of this matrix, say b_{ij}, represent the probabilities of moving from state i at the time one test is conducted to state j at the time of the next test. Furthermore, to obtain the matrix $\underline{B}$, it is necessary to define the apriori probability vector $\underline{P} = (P_0,P_1)$, where P_0 is the probability that the process mean will remain in a state of control during the testing period, and P_1 is the probability of the process mean shifting to an out-of-control state during the testing period.

Feller (7) considered the average time for an assignable cause to occur to be $1/\lambda$ hours. He also considered the probability of a shift occurring in the interval t to $t + \Delta t$ to be approximately $\lambda e^{-\lambda t}$ for small values of Δt. Based on these assumptions, Feller shows that the probability P_0 of remaining in state $\underline{u}_0$ for h hours is

$$P_0 = 1 - \int_0^h \lambda e^{-\lambda t} dt = e^{-\lambda h}$$

Since the process parameter is either in or out of control, the probability P_1 must be $P_1 = 1 - P_0 = 1 - e^{-\lambda h}$, as $P_0 + P_1 = 1$.

If R represents the production rate in units per hour, $(R=k/h)$, then P_0 becomes

$$P_0 = e^{-\lambda k/R}, \tag{2.11}$$

and P_1 becomes

$$P_1 = 1 - e^{-\lambda k/R}. \tag{2.12}$$

Assuming

$$\lambda' = \lambda/R$$

(i.e., $1/\lambda'$ is the average number of units produced before a shift occurs), P_0 becomes

$$P_0 = e^{-\lambda' k},$$

and P_1 becomes

$$P_1 = 1 - e^{-\lambda' k}.$$

Absolute definition of the transition probabilities requires the following additional assumptions: First, it is assumed that when the process goes out of control (when $\underline{u}$ shifts from $\underline{u}_0$ to $\underline{u}_1$), the process stays out of control until detected. The length of time the process is out of control lasts until H_0 is rejected. In this case, the process will not correct itself. It will also be assumed that only one shift is allowed in each testing period. The transition probability matrix $\underline{B}$ is defined as

$$\underline{B} = \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix}$$

The probability (b_{00}) of the process being in control at time t and being in control at time t+h is simply the probability P_0 of remaining in control for h hours. The probability (b_{01}) of the process being in control at time t and out of control h hours later is P_1, the probability that the process will shift out of control during that time.

The probability (b_{10}) of the process being out of control at the time t and in control at a later time is the probability (q_1) of rejecting H_0 when $\underline{u} = \underline{u}_1$, multiplied by the probability (P_0) of remaining in control for h hours.

The probability (b_{11}) of a process being out of control at time t and still out of control at time t+h is the probability (q_1) of rejecting H_0 when $\underline{u} = \underline{u}_1$ times the probability (P_1) of returning to the out-of-control state in h hours, plus the probability ($1-\underline{q}_1$) of failing to reject H_0 at time t multiplied by the probability of remaining in the out-of-control state for h hours. The probability of remaining in the out-of-control state for h hours must be one, since the null hypothesis is not rejected at time t and the process is already out of control.

From the above definitions, the transition probability matrix becomes

$$\underline{B} = \begin{bmatrix} P_0 & P_1 \\ q_1 P_0 & q_1 P_1 + 1 - q_1 \end{bmatrix}$$

According to Feller (7), since the elements b_{ij} are fixed, conditional probabilities of a process being in state j given that it was in state i the previous period, the matrix $\underline{B}$ is a stochastic matrix defining the transition probabilities of a Markov chain. If this is true, then the elements b_{ij} must satisfy the condition $0<b_{ij}<1$. This must be true because of the way in which the probability vectors $\underline{P}$ and

$\underline{q}$ are defined. Also, the sum of each row of the matrix $\underline{B}$ equals one. Therefore, the conditions

$$P_0 + P_1 = 1,$$

and

$$q_1 P_0 + q_1 P_1 + (1-q_1) = 1$$

must be true.

By definition

$$P_0 = e^{-\lambda' k},$$

and

$$P_1 = 1 - e^{-\lambda' k},$$

so obviously,

$$P_0 + P_1 = 1.$$

Since

$$P_1 = 1 - P_0,$$

the second condition can be written as

$$q_1 P_0 + q_1 - q_1 P_0 + 1 - q_1 = 1,$$

which reduces to

$$q_1P_0 - q_1P_0 + 1 = 1,$$

and the second condition is satisfied.

Thus, according to Parzen (23), $\underline{B}$ is the transition probability matrix of an irreducible positive recurrent Markov chain and there exists a vector

$$\underline{\alpha} = (\alpha_0, \alpha_1),$$

such that

$$\underline{\alpha}\underline{B} = \underline{\alpha},$$

where the elements of $\underline{\alpha}$ are the steady state unconditional probabilities of being in a particular state independent of the initial state of the process.

Rewriting the above equation will yield

$$\underline{\alpha}\underline{B} - \underline{\alpha} = \underline{0}, \qquad (2.13)$$

or

$$\underline{\alpha}(\underline{B}-\underline{I}) = \underline{0},$$

where I is a 2×2 identity matrix and $\underline{0}$ is a row vector of 2 zeroes. By adding the condition that

$$\alpha_0 + \alpha_1 = 1,$$

equation (2.13) becomes

$$\underline{\alpha}(\underline{B}-\underline{I}|1) = (\underline{0},\underline{1}), \tag{2.14}$$

where $\underline{1}$ is a 2×1 column vector of ones.

The addition of the stipulations listed above, allows us to eliminate the first column on the left side of the above equation, and the corresponding zero on the right side. Thus we have

$$\underline{\alpha B}^{*} = (\underline{0},\underline{1}), \tag{2.15}$$

where

$$B^{*} = \begin{bmatrix} b_{01} & 1 \\ b_{11}-1 & 1 \end{bmatrix}.$$

Solving for $\underline{\alpha}$ we have

$$\underline{\alpha} = (\underline{0},\underline{1})\underline{B}^{*-1}, \tag{2.16}$$

where $\underline{B}^{*-1}$ is the inverse of $\underline{B}^{*}$. Parzen (23) has shown that $\underline{\alpha}$ has a unique solution. Therefore, $\underline{B}^{*}$ must have an inverse. To determine the values of $\underline{\alpha}$ consider the Matrix $\underline{B}^{*}$ in terms of the probabilities defined previously. That is,

$$\underline{B}^* = \begin{bmatrix} P_1 & 1 \\ -q_1P_0 & 1 \end{bmatrix}$$

and the inverse is

$$\underline{B}^{*-1} = \frac{1}{P_1 + q_1P_0} \begin{bmatrix} 1 & -1 \\ q_1P_0 & P_1 \end{bmatrix};$$

thus

$$\underline{\alpha} = \left[\frac{q_1P_0}{P_1 + q_1P_0}, \quad \frac{P_1}{P_1 + q_1P_0}\right]$$

or

$$\alpha_0 = \frac{q_1P_0}{P_1 + q_1P_0}, \tag{2.17}$$

and

$$\alpha_1 = \frac{P_1}{P_1 + q_1P_0} \; . \tag{2.18}$$

The last probability vector to be defined in this section is $\underline{\gamma} = [\gamma_0, \gamma_1]$. First consider the following definition. Let F be the average fraction of the interval that elapses before the shift occurs. Duncan (5) has shown that given a shift between the n^{th} and $n+1^{st}$ sample F is given by

$$F = 1 - \frac{(1+\lambda'k)e^{-\lambda'k}}{\lambda'k(1-e^{-\lambda'k})}$$

The probability γ_0 that the process is in control ($\underline{u}=\underline{u}_0$) at any point in time is

$$\gamma_0 = \alpha_0 P_0 + F\alpha_0 P_1 \tag{2.19}$$

This equation is equal to the steady state probability that the process is in control at the beginning of the sampling period, and stays there, plus F times the probability that the process is in control at the beginning of the sampling period and shifts out of control sometime during the period. The probability γ_1 that the process is out of control ($\underline{u}=\underline{u}_1$) at any point in time is

$$\gamma_1 = \alpha_1 + (1-F)\alpha_0 P_1 \tag{2.20}$$

and is equal to the steady state probability that the process is in state ($\underline{u}=\underline{u}_1$) at the beginning of the period, plus (1-F), times the probability that the process shifts from the in control state to the out-of-control state during the period.

The sum of the probabilities of being in any state at any point in time must equal one. To verify this, consider

$$\gamma_0 + \gamma_1 = (\alpha_0 P_0 + F\alpha_0 P_1) + \alpha_1 + (1-F)\alpha_0 P_1,$$

which equals

$$\alpha_0 P_0 + \alpha_1 + \alpha_0 P_1.$$

Since

$$P_0 = 1 - P_1,$$

we have

$$\alpha_0 - \alpha_0 P_1 + \alpha_1 + \alpha_0 P_1,$$

which reduces to $\alpha_0 + \alpha_1$, which was previously defined to equal one.

Consolidated Model Review

The purpose of this section is to bring together all the components of the model developed in previous sections. For ease of reference, the key definitions will be restated.

In equation 2.1 the total expected cost was defined as

$$E(C) = E(C_1) + E(C_2) + E(C_3)$$

Using equations 2.2, 2.3, and 2.4, the above equation becomes:

$$E(C) = \frac{a_1}{k} + \frac{a_2 N}{k} + \frac{a_3}{k}[q_0\alpha_0 + q_1\alpha_1] + a_4[f_0\gamma_0 + f_1\gamma_1] \qquad (2.21)$$

In this equation, a_1 is the fixed cost per sample, a_2 is the inspection cost per unit of product, a_3 is the average cost of investigation and correcting the process, a_4 is the cost per unit associated with producing defectives, N is the number of units in a sample and k is the number of units of product produced between samples (including the units sampled).

The vector $\underline{q} = (q_0 q_1)$ is the probability of rejecting H_0 when $\underline{u}=\underline{u}_0$ and $\underline{u}=\underline{u}_1$ respectively. The vector $\underline{\alpha} = [\alpha_0, \alpha_0]$ is the steady state probability that $\underline{u}=\underline{u}_0$ and $\underline{u}=\underline{u}_1$, respectively, at the time the sample is taken. The vector $\underline{\gamma} = [\gamma_0, \gamma_1]$ is the steady state probability that $\underline{u}=\underline{u}_0$ and $\underline{u}=\underline{u}_1$, respectively, at any point in time. The vector $\underline{f}$ = (f_0, f_1) is the probability of producing a defective when $\underline{u}=\underline{u}_0$ and $\underline{u}=\underline{u}_1$, respectively.

The probabilities q_0 and q_1 were defined as follows:

$$q_0 = \int_{\frac{N-p}{p(N-1)} T^2_{\alpha,p,N-1}}^{\infty} f(F,p,N-p)dF,$$

and

$$q_1 = \int_{\frac{N-p}{p(N-1)} T^2_{\alpha,p,N-1}}^{\infty} f(F'p,N-p,\lambda)dF'.$$

The probabilities α_0 and α_1 were defined as

$$\alpha_0 = \frac{q_1 P_0}{P_1 + q_1 P_0} ,$$

and

$$\alpha_1 = \frac{P_1}{P_1 + q_1 P_0} ,$$

where

$$p_0 = e^{-\lambda' k},$$

and

$$p_1 = 1 - e^{-\lambda' k}.$$

Letting

$$K = \lambda' k, \tag{2.22}$$

then

$$p_0 = e^{-K},$$

and

$$p_1 = 1 - e^{-K}.$$

The probabilities γ_0 and γ_1 are

$$\gamma_0 = \alpha_0 p_0 + F\alpha_0 p_1,$$

and

$$\gamma_1 = \alpha_1 + (1-F)\alpha_0 p_1,$$

where

$$F = \frac{1 - (1+\lambda' k)e^{-\lambda' k}}{\lambda' k(1-e^{-\lambda' k})} .$$

Using equation 2.22, F becomes

$$F = \frac{1 - (1-K)e^{-K}}{K(1-e^{-K})} .$$

The probabilities f_0 and f_1 have not been previously defined. The probability f_0 of producing a defective given that $\underline{u}=\underline{u}_0$ depends on the process under consideration. For a given process with mean vector $\underline{u}$, upper and lower limits are specified for each component of the mean vector. These limits define the range for each mean component that will produce an acceptable unit of product.

When the process involves p quality characteristics the probability f_0 is given by

$$f_0 = 1 - \int_{a_1}^{b_1} \cdots \int_{a_p}^{b_p} \phi_0(\underline{x})dx_1 \cdots dx_p, \tag{2.23}$$

where a_i and b_i represent the lower and upper limits of the i^{th} mean, respectively. The joint density $\phi_0(\underline{x})$ of the p independent normal variates is

$$\phi_0(\underline{x}) = \frac{1 \exp[-1/2(\bar{\underline{x}}-\underline{u}_0)'\underline{V}^{-1}(\bar{\underline{x}}-\underline{u}_0)]}{(2\pi)^{1/2p}|\underline{V}|^{1/2}} .$$

The probability f_1 of producing a defective given that $\underline{u}=\underline{u}_1$ depends on the specified limits on the mean components and the magnitude of the shift of each mean component from its hypothesized value. Therefore, the probability f_1 is given by

$$f_1 = 1 - \int_{a_1}^{b_1} \cdots \int_{a_p}^{b_1} \theta_1(\underline{x})dx_1 \cdots dx_p, \tag{2.24}$$

where

$$\theta_1(\underline{x}) = \frac{\exp[-1/2(\bar{\underline{x}}-\underline{u}_1)'\underline{V}^{-1}(\bar{\underline{x}}-\underline{u}_1)]}{(2\pi)^{1/2p}|\underline{V}|^{1/2}}$$

These probabilities may be obtained from tables for $p \leq 3$, or evaluated numerically.

Applying the change of variables given in equation 2.22 to equation 2.21, the result is

$$E(C) = \frac{a_1\lambda'}{K} + \frac{a_2\lambda'N}{K} + \frac{a_3\lambda'}{K}[q_0\alpha_0+q_1\alpha_1] + a_4[f_0\gamma_0+f_1\gamma_1] \quad (2.25)$$

If the equation 2.25 is divided by a_4 and

$$A_i = a_i\lambda'/a_4$$

for i=i,2, and 3, the resulting equation is

$$E^*(C) = \frac{A_1}{K} + \frac{A_2N}{K} + \frac{A_3}{K}[q_0\alpha_0+q_1\alpha_1] + [f_0\gamma_0+f_1\gamma_1]. \quad (2.26)$$

In the expression above $E^*(C)$ is $E(C)$ divided by a_4. Since a_4 is a constant for any given problem, choosing N, K, and T^2_α to minimize $E^*(C)$ is equivalent to minimizing E(C). Thus equation 2.26 is used as the model for determining the optimal sample size (N), sampling interval (K), and critical region (T^2_α).

A Restricted Model

The model developed in this chapter is based on the assumption that the variance-covariance matrix must be estimated from the sample. The purpose of this section will be to consider what changes would result in the model if the population variance-covariance matrix were assumed to be known.

It is possible to consider the variance-covariance matrix $\underline{V}$ to be known when the same experimental variables have been under consideration for an extended period of time. When such a situation does occur, the hypothesis $H_0 : \underline{u}=\underline{u}_0$ can be treated by the test statistic

$$\chi^2 = N(\bar{\underline{x}}-\underline{u}_0)'\underline{V}^{-1}(\bar{\underline{x}}-\underline{u}_0). \tag{2.27}$$

This test statistic may be used when N observations are taken from a multivariate normal population with mean $\underline{u}$ and variance-covariance matrix $\underline{V}$. If the null hypothesis is true, χ^2 has a chi-squared distribution with p degrees of freedom. Therefore, the probability vector $\underline{q} = (q_0, q_1)$ would be defined as follows:

$$q_0 = \int_{\chi^2_{\alpha,p}}^{\infty} f(\chi^2,p)\,d\chi^2, \tag{2.28}$$

where $f(\chi^2,p)$ is the central chi-squared distribution with p degrees of freedom and is given by

$$f(\chi^2,p) = \frac{\chi^{[(p-2)/2]}\exp[-\chi/2]}{2^{p/2}\Gamma(p/2)}.$$

The probability q_1 is

$$q_1 = \int_{\chi^2_{\alpha,p}}^{\infty} f(\chi^{2'},p,\lambda)d\chi^2, \tag{2.29}$$

where $f(\chi^{2'},p,\lambda)$ is the noncentral chi-squared distribution with p degrees of freedom and noncentrality parameter λ. The noncentral chi-squared distribution is given by

$$f(\chi^{2'},p,\lambda) = \sum_{i=0}^{\infty} \frac{\chi^{[(2i+p)/2]-1}e^{-\lambda}\exp[-\chi/2]}{2^{[(2i+p)/2]}\Gamma[(2i+p)/2]i!}$$

The probabilities f_0 and f_1 are unchanged in that the equations defining f_0 and f_1 in the unrestricted model can be used in the restricted model. In both models a good estimate of the population covariance matrix $\underline{V}$ is required to evaluate the probability vector $\underline{f}$. The requirement that we have knowledge of the population covariance matrix $\underline{V}$ seems to contradict one of the basic assumptions pertinent to this investigation. This apparent contradiction is easily clarified when you consider how the population covariance matrix is used in the two models.

The restricted model requires knowledge of $\underline{V}$ to evaluate the test statistic, the noncentrality parameter and the probabilities of producing a defective. The unrestricted model required knowledge of $\underline{V}$ to evaluate the noncentrality parameter and the probabilities of producing a defective. The noncentrality parameter is used to determine the probability of rejecting the null hypothesis when a shift has occurred. In practice, the magnitude of the shift we would like to detect is arbitrarily selected. Thus, the estimate of $\underline{V}$ need not be extremely accurate. Even when the estimate of $\underline{V}$ is accurate the true value of $\underline{V}$ may change with time. The same argument can be made concerning an accurate estimate of $\underline{V}$ to determine f_0 and f_1, since the specified limits on the mean vector components are arbitrarily chosen.

The test statistic is the basis of the quality control testing procedure. Therefore, if the initial estimate of $\underline{V}$ is inaccurate, or the value of $\underline{V}$ has changed with time, using the chi-squared test statistic could result in substantial error each time the test is conducted. The T^2 test statistic defined in the unrestricted model involves estimating the population covariance matrix $\underline{V}$ from the sample each time the test is conducted. If the estimate of $\underline{V}$ from past data is inaccurate, the T^2 statistic is unaffected and the resulting quality control tests are considered more accurate.

CHAPTER III

SOLUTION METHOD

The purpose of this chapter is to formulate a method for selecting the sample size (N), the interval between samples (K), and the critical region parameter (T_α^2), which minimizes the cost function given by equation 2.26. In addition, an example of an application of the model developed in the previous chapter will be presented. The example will be limited to an investigation of a process which is characterized by two quality characteristics.

Optimization Techniques

To determine the values of N, K and T_α^2, all of which minimize the cost function (equation 2.26), implicit functions for the partial derivatives of the expected cost with respect to N, K and T_α^2 were considered. Each of these partial derivatives are functions of the inverse of the matrix $\underline{B}^*$. Since the elements of B^{*-1} are functions of N, K and T_α^2, explicit solutions for the optimal N, K and T_α^2 appear to be impossible.

To determine the nature of the cost function, numerical methods for finding the values of N, K, and T_α^2, which cause the partial derivatives to go to zero, were considered. Two major reasons exist for abandoning this approach. First, it is difficult to identify the presence of local minima since it may not be possible to solve the equations defined by the first partial derivatives. Also, the second partial derivatives must be considered to determine whether the surface of the cost function is convex. Secondly, numerical study of the original cost function is simpler. Duncan (6) obtained information about the nature of the cost function developed in his paper through computer tabulations. The result was a rectangular array that was used to draw contours, thus giving a good picture of the nature of the cost function. Duncan's work indicated that cost surfaces arising from economic control charts models were likely to have numerous local minima. In light of this it seems reasonable to employ search techniques to determine local minima for the cost model.

The method adopted for analysis of the cost function involves two stages. The first stage is used to obtain estimates of the optimal values of N, K and T_{α}^{2}. From the estimate obtained in the first stage, the second stage searches for the optimal values of N, K and T_{α}^{2}.

The calculations involved in this two-stage procedure require the use of a digital computer. In order to use a specific computer programming language[1] some changes in symbolism are necessary. These changes are listed in Appendix A. The discussion of the program in this chapter, however, is in the original model language.

The methods described in this section are generally applicable to any number of quality characteristics greater than one. The computer

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[*]UNIVAC 1108, Fortran V is used in this investigation. The program was written in the conversational mode for use with remote terminal access.

program shown in Appendix B was written for the specific application described in the numerical example.

To simplify the discussion the sample size (N), the interval between samples (K), and the critical region parameter (T_α^2), will be called the quality control test parameters or simply the test parameters. The cost coefficients A_1, A_2 and A_3 will be called simply the cost coefficients.

First Stage Procedure

The basic function of the first stage procedure is to compute the expected cost for many combinations of test parameters, cost coefficients and a number of hypothetical problems. For each combination of cost coefficients the test parameters that yield the minimum expected cost for each problem will be chosen as the preliminary estimates for the second stage procedure. A flow diagram of the procedure described above is shown in Figure 3.1.

Second Stage Procedure

The function of the second stage procedure is to determine the optimal values of the test parameters with greater accuracy than that obtained in the first stage. For each combination of cost coefficients and example problem, the preliminary estimates (call them N^*, K^* and T_α^{2*}) of the optimal test parameters were obtained in the first stage. The desired accuracy for K (call it ΔK) and the desired accuracy for T_α^2 (call it ΔT_α^2) are inputs to the second stage. The sample size accuracy is automatically set at one by the program.

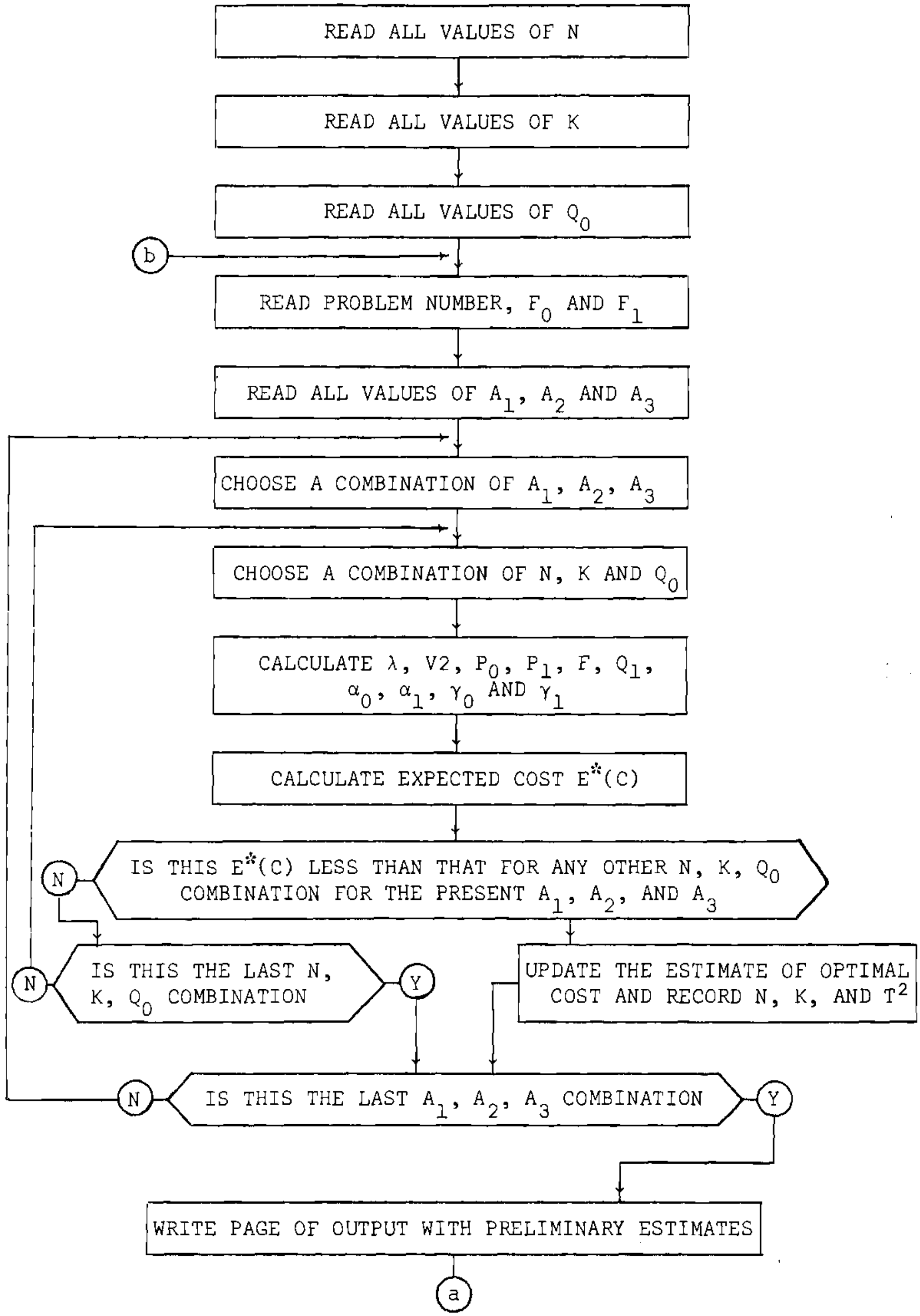

Figure 3.1 Flow Diagram of the First Stage Procedure

The expected costs are computed for all 27 combinations of N^*-1, N^*, N^*+1, $K^*-\Delta K$, K^*, $K^*+\Delta K$ and $T^2_\alpha-\Delta T^2_\alpha$, T^2_α, $T^2_\alpha+\Delta T^2_\alpha$. If the expected cost for N^*, K^*, and T^2_α is not less than or equal to all other expected costs, a new value of N^*, K^*, and T^2_α is obtained by using the parameters yielding a minimum expected cost. The process is repeated until N^*, K^*, and T^2_α yields the minimum of all 27 combinations. A flow diagram of the procedure described above is shown in Figure 3.2.

A Hypothetical Example

In this section an example of an application of the model developed in Chapter II is presented. The example will be restricted to a process characterized by two quality characteristics. The process parameter is the mean vector $\underline{u}$. The test statistic is T^2 and is distributed $T^2_{p,N-1}$. The quality measurement $\underline{x}$ is assumed to have a bivariate normal distribution with mean vector $\underline{u}$ and covariance matrix $\underline{V}$, thus $\bar{\underline{x}}$ has a bivariate normal distribution with mean vector $\underline{u}$ and covariance matrix $\underline{V}/N$.

The example was formulated by arbitrarily selecting values for $\underline{u}_0$, $\underline{\delta}$, $\underline{\rho}$ and $\underline{V}$. The values chosen for the mean vector when the null hypothesis is true are

$$\underline{u}_0 = \begin{bmatrix} 50 \\ 60 \end{bmatrix} .$$

If we define $\underline{\delta}$ as the shift parameter, then $\underline{\delta}$ represents the magnitude of the shift of the mean vector that we would like to detect. The values chosen are

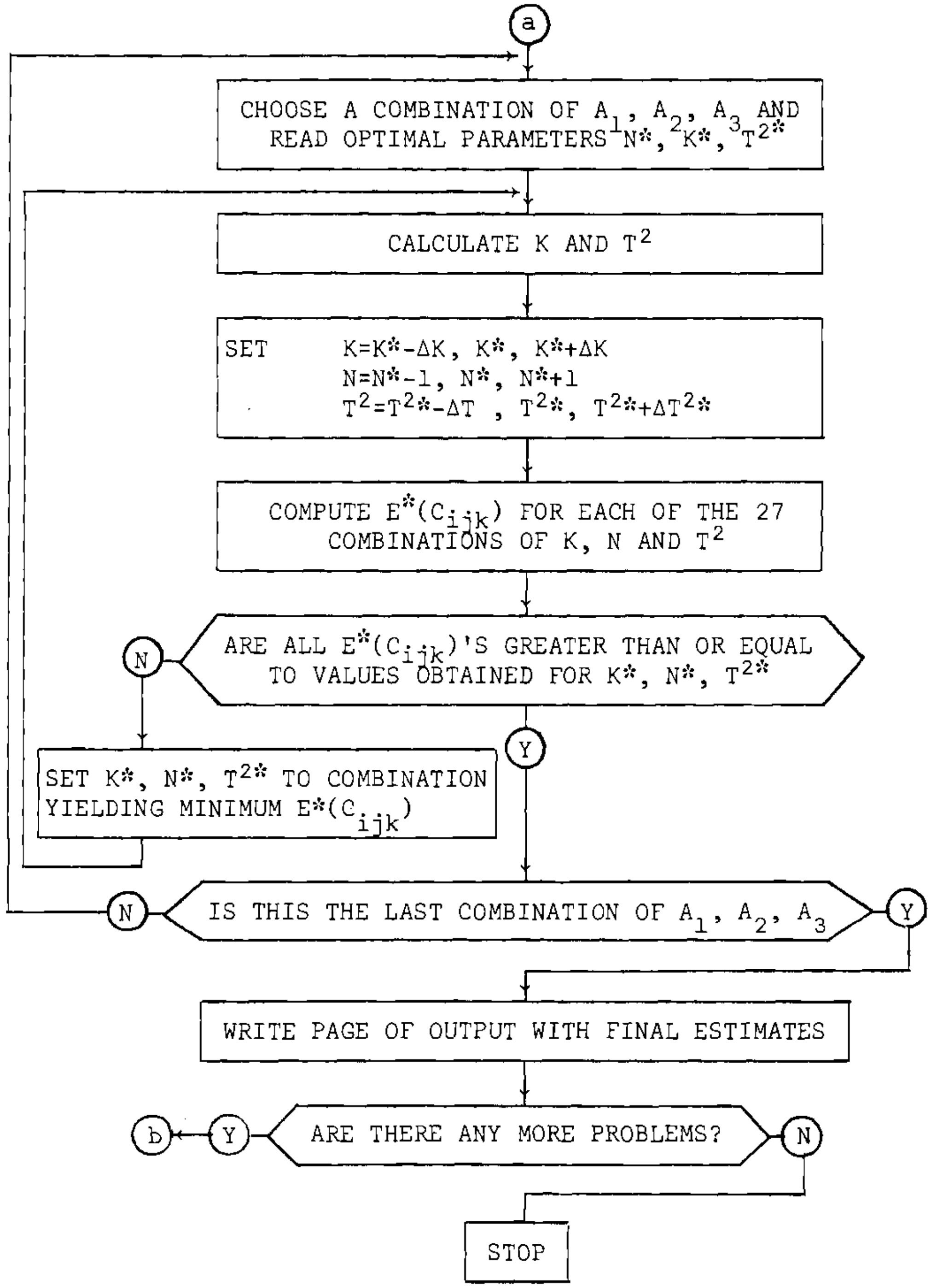

Figure 3.2 Flow Diagram of the Second Stage Procedure

$$\underline{\delta} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

. Defining $\underline{\rho}$ as the specification on the mean vector, then $\underline{u}_0 \pm \underline{\rho}$ represents the range in which the process produces nondefective units of product. The values chosen for $\underline{\rho}$ are

$$\underline{\rho} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

The estimate of the covariance matrix in the vibariate case is given by

$$\underline{\hat{V}} = \begin{bmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{21} & \hat{\sigma}_{22} \end{bmatrix},$$

where the diagonal elements represent estimates of the variances and the off diagonal elements are estimates of covariances based on past data. By varying the components of $\underline{\hat{V}}$, four problems were specified in an attempt to analyze the cost model under different conditions. The four estimated covariance structures selected are as follows:

$$\underline{\hat{V}}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\underline{\hat{V}}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2.5 \end{bmatrix}$$

$$\hat{\underset{\sim}{V}}_3 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\hat{\underset{\sim}{V}}_4 = \begin{bmatrix} 2 & 1 \\ 1 & 2.5 \end{bmatrix}$$

Calculation of the Vector q

The probability of rejecting H_0 when $\underline{u}=\underline{u}_0$ is q_0 and the probability of rejecting H_0 when $\underline{u}=\underline{u}_1$ is q_1. The probabilities q_0 and q_1 were previously defined as

$$q_0 = \int_{\frac{N-p}{p(N-1)} T^2_{\alpha,p,N-1}}^{\infty} f(F,p,N-p)dF,$$

and

$$q_1 = \int_{\frac{N-p}{p(N-1)} T^2_{\alpha,p,N-1}}^{\infty} f(F',p,N-p,\lambda)dF',$$

with frequency functions

$$f(F,p,N-p) = \frac{\Gamma(N/2)[p/(N-p)]^{p/2}F^{(p-2)/2}}{\Gamma(p/2)\Gamma[(N-p)/2][1+pF/(N-p)]^{N/2}}$$

and

$$f(F',p,N-p,\lambda) = \sum_{i=0}^{\infty} \frac{\Gamma[(2i+N)/2][p/(N-p)]^{(2i+p)/2}\lambda^i e^{-\lambda}F^{(2i+p-2)/2}}{\Gamma[(N-p)/2]\Gamma[(2i+p)/2]i![1+pF/(N-p)]^{(2i+N)/2}}$$

To calculate the probabilities q_0 and q_1 with a digital computer approximations for the above integrals were used. The approximations are as follows:

$$q_i = \int_x^\infty \frac{1}{(2\pi)^{1/2}} e^{-Z^2/2}\, dz \qquad (3.1)$$

where the value of q_0 is found by letting x in equation 3.1 equal

$$x = \frac{[v_2T^2/v_1(N-1)]^{1/3}[1-2/9v_2] - [1-2/9v_1]}{[2/9v_1+2/9v_2[v_2T^2/v_1(N-1)]^{2/3}]^{1/2}}, \qquad (3.2)$$

and the value of q_1 is found by letting x in equation 3.1 equal

$$x = \frac{[v_2T^2/(v_1+\lambda)(N-1)]^{1/3}[1-2/9v_2] - [1-2(v_1+2\lambda)/9(v_1+\lambda)^2]}{[2(v_1+2\lambda)/9(v_1+\lambda)^2 + 2/9v_2[v_1T^2/(v_1+\lambda)(N-1)]^{2/3}]^{1/2}}. \qquad (3.3)$$

To simplify equations 3.2 and 3.3 the degrees of freedom p and N-p are replaced with v_1 and v_2, respectively. The normal approximation for q_0 was developed by Paulson (24). The approximation for q_1 is an extension of Paulson's approximation and the derivation can be found in Appendix C.

Hastings' (9) approximation of the normal integral was used in conjunction with the above approximations. He found that

$$\int_0^x \frac{2}{(\pi)^{1/2}} e^{-t^2/2}\, dt$$

can be approximated by

$$1 - \frac{1}{[1+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6]^{16}} \qquad (3.4)$$

where the values for a_i are as follows:

$$a_1 = .0705230784$$

$$a_2 = .0422820123$$

$$a_3 = .0092705272$$

$$a_4 = .0001520143$$

$$a_5 = .0002765672$$

$$a_6 = .0000430638$$

From equation 3.4, it can be shown that equation 3.1 becomes

$$q_i \doteq \frac{.5}{[1+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6]^{16}}, \qquad (3.5)$$

where the value for x in equation 3.2 is used to find q_0 and the value for x in equation 3.3 is used to find q_1. The equation

$$c_i = a_i/2^{i/2}$$

is used to find the following values:

$$c_1 = .0498673470$$

$$c_2 = .0211410062$$

$$c_3 = .0032776263$$

$$c_4 = .0000380036$$

$$c_5 = .0000488900$$

$$c_6 = .0000053830$$

Hastings' approximation of the normal integral has an accuracy of .000003 and the approximations involving x are accurate to within .005 of the true value.

In the example v_1 is set equal to two since the process is assumed to have only two quality characteristics. The noncentrality parameter is given by

$$\lambda = N(\underline{u}_1-\underline{u}_0)'\hat{\underline{V}}^{-1}(\underline{u}_1-\underline{u}_0). \tag{3.6}$$

Recall that $\underline{\delta}$ was defined as the magnitude of the shift in the mean vector. Thus,

$$\underline{\delta} = \underline{u}_1 - \underline{u}_0,$$

and the noncentrality parameter becomes

$$\lambda = N(\underline{\delta}'\underline{\hat{V}}^{-1}\underline{\delta}).$$

For the example, a further simplification was made such that

$$\pi = (\underline{\delta}'\underline{\hat{V}}^{-1}\underline{\delta}),$$

thus

$$\lambda = N\pi,$$

where the values of π were inputs for each problem. The values of π are shown in Table 1.

Table 1. Values of π for the Four Example Problems

	Problem Number			
	1	2	3	4
π	15.25	26.9	20.67	18.625

Calculation of the Vector $\underline{f}$

The probability of producing a defective given that $\underline{u}=\underline{u}_0$ is f_0 and the probability of producing a defective given that $\underline{u}=\underline{u}_1$ is f_1. The probabilities f_0 and f_1 were previously defined as

$$f_0 = 1 - \int_{a_1}^{b_1} \cdots \int_{a_p}^{b_p} \phi_0(x)dx_1 \cdots dx_p,$$

and

$$f_1 = 1 - \int_{a_1}^{b_1} \cdots \int_{a_p}^{b_p} \theta_1(x)dx_1 \cdots dx_p.$$

In the bivariate case f_0 and f_1 become

$$f_0 = 1 - \int_{a_1}^{b_1} \int_{a_2}^{b_2} \phi_0(\underline{x})dx_1dx_2,$$

and

$$f_1 = 1 - \int_{a_1}^{b_1} \int_{a_2}^{b_2} \theta_1(\underline{x})dx_1dx_2,$$

where the frequency functions are

$$\phi_0(\underline{x}) = \frac{\exp[-1/2(\bar{\underline{x}}-\underline{u}_0)'\hat{\underline{V}}^{-1}(\bar{\underline{x}}-\underline{u}_0)}{(2\pi)^{1/4}|\hat{\underline{V}}|^{1/2}}$$

and

$$\theta_1(\underline{x}) = \frac{\exp[-1/2(\bar{\underline{x}}-\underline{u}_1)'\hat{\underline{V}}^{-1}(\bar{\underline{x}}-\underline{u}_1)}{(2\pi)^{1/4}|\hat{\underline{V}}|^{1/2}} .$$

A defective is defined as any product whose measurement $\underline{x}$ lies outside the interval

$$(\underline{u}_0+\underline{\rho},\ \underline{u}_0-\underline{\rho}).$$

To determine the values, f_0 and f_1 tables of the bivariate normal distribution function (26) were used. In application of the tables, the variables are first standardized, that is, the transformations required for f_0 are

$$-h = \frac{a_1 - u_{0x_1}}{\sigma_{x_1}}, \quad h = \frac{b_1 - u_{0x_1}}{\sigma_{x_1}}$$

and

$$-k = \frac{a_2 - u_{0x_2}}{\sigma_{x_2}}, \quad k = \frac{b_2 - u_{0x_2}}{\sigma_{x_2}}$$

To find f_1, the transformations are

$$-h = \frac{a_1 - u_{1x_1}}{\sigma_{x_1}}, \quad h = \frac{b_1 - u_{1x_1}}{\sigma_{x_1}}$$

and

$$-k = \frac{a_2 - u_{1x_2}}{\sigma_{x_2}}, \quad k = \frac{b_2 - u_{1x_2}}{\sigma_{x_2}},$$

where

$$u_{1x_1} = u_{0x_1} + \delta_{x_1}$$

and

$$u_{1x_2} = u_{0x_2} + \delta_{x_2}.$$

The values for a_1, b_1, a_2, and b_2 in the above equations are as follows:

$$a_1 = u_{0x_1} - \rho_{x_1}$$

$$b_1 = u_{0x_1} + \rho_{x_1}$$

$$a_2 = u_{0x_2} - \rho_{x_2}$$

$$b_2 = u_{0x_2} + \rho_{x_2}$$

The vector $\underline{f}$ of probabilities of a defective for the four problems were inputs in the first stage computations and are shown in Table 2.

Table 2. The Vector $\underline{f}$ of Probabilities of a Defective

	Problem Number			
$\underline{f}$	1	2	3	4
f_0	.002072	.004988	.0081124	.0127296
f_1	.581751	.545065	.577177	.596009

First Stage Computations

The function of the first stage of the computational procedure is to provide a means of studying the general behavior of the model and to provide preliminary estimates of the optimal values of N, K, and T_α^2 for use in the second stage. This function is accomplished by evaluating the expected cost for many combinations of test parameters, cost coefficients and example problems. Therefore, the expected cost was computed for the four problems previously specified. The values of the test parameter (N) were chosen on the basis of intuitive judgment concerning the most likely range of optimal values. Sample sizes (N) of 3, 5, 7, and 9 were chosen.

The selection of the values for the critical region parameter (T_α^2) was somewhat more difficult. Recall that the probability q_0 depends on the sample size and the T_α^2 value. Therefore, a different T_α^2 value for each N is required to specify a particular value of q_0. In light of this fact it seemed reasonable to input values of q equal to .001, .01, .1, and .25. When one of the values of q_0 is selected the computer calculates the T_α^2 value and prints the value along with the values of N and K that are preliminary optimals for a set of cost coefficients.

To identify a reasonable range of K, recall that

$$K = \lambda' k,$$

where k is the number of units produced between samples, and is a measure of the frequency of process shifts, since $1/\lambda'$ is the average number of units produced before the process goes out of control after it has been returned to the control state. A value of K equal to one indicates a frequency of sampling equal to the frequency of process shifts. Thus values of K greater than one indicate rather infrequent sampling, while small values of K (say K=.01) indicate very frequent sampling. Values of K between .01 and 1 would appear to include the reasonable range.

Thus the values .01, .02, .03, .05, .07, .10, .20, .30, .50, .70, 1.0, and 5.0 were chosen.

The choice of values for the cost coefficients is a little more complicated. Recall, for example, the definition of A_1

$$A_1 = \frac{a_1 \lambda'}{a_4} ,$$

where a_1 is the cost of taking a sample and a_4 is the cost associated with producing a defective unit. A_2 and A_3 are similarly defined. It is difficult to determine intuitively the practical range of each A_i since each is composed of three variables. For certain combinations of cost coefficients the optimal value of the interval between samples (K) could be expected to indicate that a quality control procedure is unwarranted (i.e., a very large value for the optimal K). For other combinations of cost coefficients, the optimal value of K could be expected to indicate 100 per cent inspection (i.e., very small value for the optimal K).

To identify the practical region for the cost coefficients, several runs of the first stage procedure were carried out. Preliminary estimates of the optimal values of K were obtained and the largest value of each of the cost coefficients, for which the optimal value of K was less than one, was identified. These values were chosen as the maximum values of each cost coefficient. Other values were shown as .1 and .01 times the maximum value. The values of the cost coefficients selected for study are shown in Table 3.

Table 3. Values of Cost Coefficients Selected for Study

Values of Cost Coefficients		
A_1	A_2	A_3
.0001	.00001	.001
.001	.0001	.01
.01	.001	.1

Using the computer program shown in Appendix B, preliminary estimates of the optimal values of N, K, and T_α^2, along with the associated expected costs, were obtained for problems one through four. The results are tabulated in Appendix D.

Second Stage Computations

The second stage requires that preliminary estimates of the optimal values of the test parameters (N^*,K^* and T_α^{2*}), as well as the desired accuracy for K(ΔK) and $T_\alpha^2(\Delta T_\alpha^2)$, be supplied as input data. The expected costs are computed for all 27 combinations of N^*-1, N^*, N^*+1, $K^*-\Delta K$, K^*, $K^*+\Delta K$ and $T_\alpha^{2*}-\Delta T_\alpha^2$, T_α^{2*}, $T_\alpha^{2*}+\Delta T_\alpha^2$. The search procedure involves starting with the point (N^*,K^*,T_α^{2*}) and calculating the expected costs for the other 26 combinations. If the starting point is thought of as being in the center of a cube, then the remaining 26 points are the 8 corner points, the center of the 12 lines defining the cube, and the points located at the center of the 6 surfaces of the cube.

The results of this search procedure depend on the step size (i.e., ΔT_α^2 and ΔK). Therefore, after several preliminary calculations, ΔK and ΔT_α^2 were set equal to 0.01 of the current value of K and T_α^2. The above values for ΔK and ΔT_α^2 were deemed adequate, since a change of one ΔK or ΔT_α^2, in the region of the optimal K and T_α^2, produced less than a 1 per cent change in the expected cost.

The results also depend on the starting point since the cost function may not be strictly convex or even convex and the search procedure will locate only a local minimum. However, the results obtained for the example problem, while most likely only local minima, do fall in a realistic region of the surface so far as practical operating conditions are concerned.

Analysis of Results

The general model developed in Chapter II and the general solution method presented in this chapter were used to obtain optimal test parameters for the tests of hypothesis about the mean vector of a process. The optimal sample size (N), time between samples (K) and critical region parameter (T_α^2) along with the minimum expected cost $E^*(C)$ are shown in Appendix E for selected values of the cost coefficients (A_1, A_2 and A_3) and four example problems. The effect of each cost coefficient and each problem on the optimal values of the test parameters is discussed below.

Effect of $\hat{V}$ of Optimal $E^*(C)$, N, K and T_α^2

Recall that the four problems were specified by varying the components of the covariance matrix $\underline{\hat{V}}$. The estimate $\underline{\hat{V}}$ is used to evaluate the noncentrality parameter (λ) and the probabilities of producing a defective ($\underline{f}$). Since the estimate may not be extremely accurate, it is important to determine the models sensitivity to variations in $\underline{\hat{V}}$. The effect of the four problems on the values of $E^*(C)$, N, K and T^2_α is shown in Table 4 for one value of A_1, A_2, and A_3.

Table 4. The Effect of $\hat{V}$ on the Optimal Values of $E^*(C)$, N, K and T^2_α

$A_1 = 0.001$; $A_2 = 0.0001$; $A_3 = 0.01$.

	Optimal Costs and Test Parameters			
Problem	$E^*(C)$	N	K	T^2
1	0.06202	10.0	0.08	17.67
2	0.06092	8.0	0.09	21.82
3	0.06623	9.0	0.09	19.67
4	0.07191	9.0	0.08	17.67

From Table 4 it is seen that variations in $\underline{\hat{V}}$ have little or no effect on the expected cost $E^*(C)$, the optimal sample size (N), the optimal time between samples (K) and the optimal critical region parameter T^2_α. Therefore, if the estimate of the true value of the covariance matrix is not accurate, the model can still be expected to produce reasonable results.

<u>Effect of A_1 on the Optimal Test Parameters</u>

The effect of the fixed costs associated with taking a sample and making the test (A_1) on the optimal values of the test parameters is shown in Table 5 for one value of A_2 and A_3 and the four example problems.

Table 5. The Effect of A_1 on the Optimal Values of N, K and T^2_α

A_2 = 0.0001 and A_3 = 0.01

Problem	Cost A_1	Optimal Test Parameters N	K	T^2
1	0.0001	10.0	0.07	17.67
	0.001	10.0	0.08	17.67
	0.01	10.0	0.22	14.51
2	0.0001	8.0	0.06	24.0
	0.001	8.0	0.09	21.82
	0.01	9.0	0.22	24.0
3	0.0001	9.0	0.06	21.82
	0.001	9.0	0.09	19.64
	0.01	9.0	0.22	17.67
4	0.0001	9.0	0.06	19.64
	0.001	9.0	0.08	17.67
	0.01	10.0	0.21	19.45

The effect of the cost A_1 on the optimal critical region parameter (T^2_α) is highly dependent upon the values of the other cost coefficients. No useful generalizations are apparent.

As A_1 increases there is a slight tendency for the optimal value of N to increase. This increase is more apparent when the cost of inspecting a unit (A_2) is small.

The effect of A_1 on the optimal time between samples (K) is very apparent. As expected, an increase in the fixed cost of taking a sample and making the test increases the optimal interval between samples.

Effect of A_2 on the Optimal Test Parameters

The effect of the cost of inspecting a unit of product (A_2) is shown in Table 6 for one value of A_1 and A_3 and the four example problems.

Table 6. The Effect of A_2 on the Optimal Values of N, K and T^2_α

$A_1 = 0.001$ and $A_3 = 0.01$

Problem	Cost A_2	Optimal Test Parameters N	K	T^2
1	0.00001	14.0	0.07	28.75
	0.0001	10.0	0.08	17.67
	0.001	7.0	0.20	8.160
2	0.00001	11.0	0.06	35.14
	0.0001	8.0	0.09	21.82
	0.001	5.0	0.18	8.900
3	0.00001	12.0	0.06	31.95
	0.0001	9.0	0.09	19.64
	0.001	5.0	0.18	6.690
4	0.00001	13.0	0.06	31.95
	0.0001	9.0	0.08	17.67
	0.001	6.0	0.18	8.080

It is apparent that the optimal value of N decreases, as expected, with increases in A_2.

As A_2 increases, Table 6 indicates that the optimal values of the critical region parameter (T^2_α) decreases. This decrease is probably associated with the corresponding decrease in the optimal sample size. By decreasing T^2_α when N decreases, the probability of a Type II error is stabilized.

The cost A_2 has a significant effect on the optimal interval between samples. When the cost of inspecting a unit is small, more frequent sampling is justified. Thus increases in A_2 cause the optimal value of K to increase.

Effect of A_3 on the Optimal Test Parameters

The effect of the cost of investigating and correcting the process (A_3) on the optimal values of the test parameters is shown in Table 7 for one value of A_1 and A_2 and the four example problems.

Table 7. The Effect of A_3 on the Optimal Values of N, K and T^2_α

$A_1 = 0.001$ and $A_2 = 0.0001$

Problem	Cost A_3	Optimal Test Parameters N	K	T^2
1	0.001	7.0	0.08	8.160
	0.01	10.0	0.08	17.67
	0.1	13.0	0.09	29.04
2	0.001	5.0	0.08	8.090
	0.01	8.0	0.09	21.82
	0.1	11.0	0.08	42.52
3	0.001	5.0	0.08	6.080
	0.01	9.0	0.09	19.64
	0.1	12.0	0.08	35.14
4	0.001	6.0	0.08	7.350
	0.01	9.0	0.08	17.67
	0.1	12.0	0.08	31.95

The cost of investigating and correcting the process (A_3) does not affect the optimal time between samples (K) to any significant degree.

As the cost A_3 increases, decreases in the optimal probability of falsely rejecting H_0 might be expected. Table 7 indicates such a tendency with increases in the optimal critical region parameter (T_α^2) and the optimal sample size (N). Thus, in most cases, the decreases in Type I error is accomplished by increasing T_α^2.

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

In the development of the general model, it was assumed that when the process parameter shifts to an out-of-control state it will not shift back in control until the shift is detected by a value of the test statistic falling in the critical region. It was further assumed that if the shift is not detected by the first sample taken after the shift occurs, the process may not shift to a worse state. That is, the process may not shift to a worse state nor will it improve. It is recognized that this simplified model does not accurately describe the real world behavior; however, it was hoped that the departure from reality would not affect the optimal values of N, K, and T^2_α. In a recent article, Duncan (6) showed that when there is more than one assignable cause, the model approaches reality but only the local-minimum solutions remain. Furthermore, the solutions obtained can be approximated by solutions of single-cause models. Therefore, the model presented in this thesis considered only one out-of-control state, and this may be adequate in many applications.

A two-stage numerical procedure was developed and programmed for a digital computer to determine the optimal sample size (N), the optimal interval between samples (K), and the optimal critical region parameter (T^2_α).

The two-stage procedure was used to determine the optimal values of N, K, and T_{α}^{2} for quality control tests of the mean vector of a bivariate normal process. The optimal test parameter values are tabulated for various cost and example problems.

The results indicate that variations in the estimate of the covariance matrix ($\hat{\underline{V}}$) have little or no effect on the optimal values of $E^{*}(C)$, N, K and T_{α}^{2}.

The optimal value of N is significantly increased by decreases in A_2 and increases in A_3. With small values of A_2, additional increases in the optimal value of N occur with increases in A_1.

The optimal value of K is greatly increased by increases in A_1 and to a lesser degree increased by increases in A_2. The value of A_3 has virtually no effect on the optimal value of K.

As the value of A_3 increases, the optimal value of T_{α}^{2} greatly increases. The same is true for decreases in A_2. The effect of A_1 on the optimal value of T_{α}^{2} depends on the values of A_2 and A_3.

Recommendations

As a result of this investigation several additional research topics may be proposed. First, it would be of interest to extend the present model to consider more than one out-of-control state. Recent work by Duncan (6) has indicated that in the univariate case a single assignable cause model is sufficient, but it is not known if this holds in the multivariate case.

Second, the Markovian assumption regarding out-of-control states should be investigated. While this assumption is reasonable in many processes, Baker (3) has shown that univariate quality control models are sensitive to this assumption. He considers the number of periods the process remains in control to be distributed geometrically, and shows that if the Markovian assumption is made, erroneous results can be obtained. This investigation could be extended to the multivariate case.

Third, the optimization technique used in this investigation is perhaps not the most efficient available. It is recommended that various optimization techniques be tried to improve the efficiency of the solution method.

Fourth, the model developed in this investigation requires that a shift in the process mean be specified. It would be of considerable interest to investigate the sensitivity of the model to the shift specification. If the model is insensitive, then the manner of the shift specification would not be too critical.

Finally, the nature of the cost surface should be investigated further, to determine its approximate contours and the location of the local minima. It may be that local minima always lie in reasonable regions, so far as the control parameters are concerned.

APPENDICES

APPENDIX A

CONVERSION OF MODEL TO COMPUTER LANGUAGE

Model Symbol	Computer Symbol
N	SN
K	SK
ΔK	DELK
π	PI
A_1	A1
A_2	A2
A_3	A3
F	F
λ	LAMBA
P_0	P0
P_1	P1
f_0	F0
f_1	F1
q_0	Q0
q_1	Q1
T^2	TSQ
ΔT^2	DELTSQ
α_0	ALPHA0
α_1	ALPHA1
γ_0	GAMA0
γ_1	GAMA1
$E^*(C)$	C
MIN $E^*(C)$	CMIN
K^*	SK0

Model Symbol	Computer Symbol
N^*	SNO
T^{2*}	TSQO
v_1	V2
v_2	V2

APPENDIX B

FORTRAN PROGRAM FOR THE OPTIMIZATION PROCEDURE

```
    1 FORMAT(I2)
    2 FORMAT( )
    3 FORMAT(' FOR PI =',F15.8,' ENTER VALUE FOR A1, A2, AND A3')
    5 FORMAT(' ENTER',I3,' VALUES IN FREE FIELD FORMAT')
    6 FORMAT(' HOW MANY VALUES OF N  (I2 FORMAT)')
    7 FORMAT(' ENTER',I3,' VALUES EACH CONTAINING A (.) AND '
     *'FOLLOWED BY A (,)')
    8 FORMAT(' HOW MANY VALUES OF K  (I2 FORMAT)')
    9 FORMAT(' ENTER',I3,' VALUES EACH CONTAINING A . AND '
     *'ENDING FOLLOWED BY A , ')
   10 FORMAT(' HOW MANY VALUES OF Q0  (I2 FORMAT)')
   11 FORMAT(' ENTER',I3,' VALUES IN FREE FIELD FORMAT')
   12 FORMAT(' ENTER T SQUARE ARRAY 4X4 EACH ROW ON A CARD')
   13 FORMAT(' PLELIMINARY ESTIMATES FOR PROBLEM', I3)
  113 FORMAT(' OPTIMAL ESTIMATES FOR PROBLEM',  I3)
   14 FORMAT(34(' *'),/,'    COST  *    COST  * PARAMETER *',14X'COST A1'
     1,14X,'*',/,4X,'A2',3X,'*     A3    *',11X,'*',3(F9.7,'  *'))
   15 FORMAT(F8.7,' *',9X,'*',4(11X,'*'))
   16 FORMAT(9X,'*',F8.6,' *    E*(C)    *',3(F9.5,'  *'))
   17 FORMAT(9X,'*',9X,'*',5X,'N',5X,'*',3(F7.1,4X,'*'))
   18 FORMAT(9X,'*',9X,'*',5X,'K',5X,'*',3(F7.2,4X,'*'))
   19 FORMAT(9X,'*',9X,'*    TSQR     *',F8.2,3X,'*',2(F8.2,3X,'*'))
   20 FORMAT(34(' *'))
   21 FORMAT(9X,30('*  '))
   22 FORMAT('  ')
      REAL LAMB
      DIMENSION Q0(4),SN(4),SK(11),A1(3),A2(3)
     1,A3(3),CMIN(3,3,3),SNM(3,3,3),SKM(3,3,3),TSQM(3,3,3)
     2,COS(3,3,3)
      DEFINE TSQ(Y,X)=((1.0/Y)**(2.0/(X-2.0))-1.0)*(X-1.0)
      DEFINE DEL(X)=.1*X
      C1=.0498673470
      C2=.0211410062
      C3=.32776263E-2
      C4=.380036E-4
      C5=.488906E-4
      C6=.5383E-5
      V1=2
C READ NUMBER OF N'S AND N VALUES
      WRITE(6,6)
      READ (5,1) NSN
      WRITE(6,7) NSN
      READ(5,2) (SN(I),I=1,NSN)
C READ NUMBER OF K'S AND K VALUES
      WRITE(6,8)
      READ(5,1) NSK
      WRITE(6,7) NSK
      READ(5,2) (SK(I),I=1,NSK)
C READ NUMBER OF Q0'S AND Q0 VALUES
      WRITE(6,10)
      READ(5,1) NQ0
      WRITE(6,7) NQ0
      READ(5,2) (Q0(I),I=1,NQ0)
C CHOOSE A PI VALUE AND LOOP THROUGH A SET OF OUTPUT
```

```
C
  301 WRITE(6,30)
      READ(5,2,END=501) PN,PI,F0,F1
      NP=PN
   30 FORMAT(' ENTER VALUES FOR THE PROBLEM NUMBER,PI,F0 AND F1')
      WRITE (6,3) PI
      READ(5,2) (A1(I),I=1,3)
      READ(5,2) (A2(I),I=1,3)
      READ(5,2) (A3(I),I=1,3)
C INITIALIZE THE COSTMIN MATRIX FIRST
      DO 100 K=1,3
      DO 100 L=1,3
      DO 100 M=1,3
      CMIN(K,L,M)=10.0**35
C NEXT, FOR FIXED VALUES OF A'S TRY ALL N, K, AND Q0 VALUES
      DO 75 JN=1,NSN
      LAMB=PI*SN(JN)
      V2=SN(JN)-2.0
      DO 75 JK=1,NSK
      P0=EXP(-SK(JK))
      P1=1.0-P0
      F=1.0/SK(JK)-P0/P1
      DO 75 JQ=1,NQ0
      V1PL=V1+LAMB
      V2TSQ=V2*TSQ(Q0(JQ),SN(JN))
      X1=(V2TSQ/(V1PL*(SN(JN)-1.0)))**(0.3333)*(1.-2.0/(9.0*V2))
      X2=1.0-2.0*(V1+2.0*LAMB)/(9.0*(V1+LAMB)**2)
      X3=2.0*(V1+2.0*LAMB)/(9.0*(V1+LAMB)**2)+2.0/(9.0*V2)
     **(V2TSQ/((V1+LAMB)*(SN(JN)-1.0)))**(0.66667)
      X=(X1-X2)/SQRT(X3)
      Q1=(0.5)/(1.0+C1*X+C2*X**2+C3*X**3+C4*X**4+C5*X**5+C6*X**6)**16
      ALPHA0=(Q1*P0)-(P1+Q1*P0)
      ALPHA1=P1/(P1+Q1*P0)
      GAMA0=ALPHA0*P0+ALPHA0*F*P1
      GAMA1=ALPHA1+(1.0-F)*ALPHA0*P1
      V2TSQ=V2*TSQ(F0,SN(JN))
      C=(A1(K)+A2(L)*SN(JN)+A3(M)*(Q0(JQ)*ALPHA0+Q1*ALPHA1))
     1/SK(JK)+F0*GAMA0+F1*GAMA1
      IF (C.GT.CMIN(K,L,M)) GO TO 75
      CMIN(K,L,M)=C
      SNM(K,L,M)=SN(JN)
      SKM(K,L,M)=SK(JK)
      TSQM(K,L,M)=TSQ(Q0(JQ),SN(JN))
   75 CONTINUE
  100 CONTINUE
C
C WRITE A PAGE OF OUTPUT
C
      WRITE(6,13) NP
      WRITE(6,14) (A1(I),I=1,3)
      WRITE(6,20)
      DO 300 J=1,3
      WRITE(6,15) A2(J)
      DO 200 K=1,3
```

```
      WRITE(6,16) A3(K),(CMIN(I,J,K),I=1,3)
      WRITE(6,17) (SNM(I,J,K),I=1,3)
      WRITE(6,18) (SKM(I,J,K),I=1,3)
      WRITE(6,19) (TSQM(I,J,K),I=1,3)
      IF (K.NE.3) WRITE(6,21)
  200 CONTINUE
      WRITE(6,20)
  300 CONTINUE
      DO 2000 I=1,3
      DO 2000 J=1,3
      DO 2000 K=1,3
      DELTSQ=DEL(TSQM(I,J,K))
      DELK=DEL(SKM(I,J,K))
      TSQO=TSQM(I,J,K)
      TSQD=TSQO-2.0*DELTSQ
      SNO=SNM(I,J,K)
      SNDL=SNO-2.0
      SKO=SKM(I,J,K)
      SKDL=SKO-2.0*DELK
      MIKE=0
 1000 CONTINUE
      DO 1001 KTSQD=1,3
      TSQD=TSQD+DELTSQ
      DO 1001 KND=1,3
      IF (KND.EQ.1) SND=SNDL
      SND=SND+1.0
      DO 1001 KKD=1,3
      IF (KKD.EQ.1) SKD=SKDL
      SKD=SKD+DELK
      IF (SND.LT.2.9) SND=3.0
      LAMB=PI*SND
      V2=SND-2.0
      PO=EXP(-SKD)
      P1=1.0-PO
      F=1.0/SKD-PO/P1
      V1PL=V1+LAMB
      V2TSQ=V2*TSQD
      X1=(V2TSQ/(V1PL*(SND-1.0)))**(0.3333)*(1.-2.0/(9.0*V2))
      X2=1.0-2.0*(V1+2.0*LAMB)/(9.0*(V1+LAMB)**2)
      X3=2.0*(V1+2.0*LAMB)/(9.0*(V1+LAMB)**2)+2.0/(9.0*V2)
     **(V2TSQ/((V1+LAMB)*(SND-1.0)))**(0.66667)
      X=(X1-X2)/SQRT(X3)
      Q1=(0.5)/(1.0+C1*X+C2*X**2+C3*X**3+C4*X**4+C5*X**5+C6*X**6)**16
      X1=(V2TSQ/(V1*(SND-1.0)))**(0.3333)*(1.0-(2./(9.*V2)))
      X2=1.0-(2./(9.*V1))
      X3=2./(9.*V1)+(2./(9.*V2))*(V2TSQ/(V1*(SND-1.)))**(.6667)
      X=(X1-X2)/SQRT(X3)
      QOD=.5/(1.+C1*X+C2*X**2+C3*X**3+C4*X**4+C5*X**5+C6*X**6)**16
      ALPHAO=(Q1*PO)/(P1+Q1*PO)
      ALPHA1=P1/(P1+Q1*PO)
      GAMAO=ALPHAO*PO+ALPHAO*F*P1
      GAMA1=ALPHA1+(1.0-F)*ALPHAO*P1
      V2TSQ=V2*TSQ(FO,SND)
      C=(A1(I)+A2(J)*SND+A3(K)*(QOD*ALPHAO+Q1*ALPHA1))
```

```
     1/SKD+FO*GAMA0+F1*GAMA1
 1001 COS(KTSQD,KND,KKD)=C
      C=10.**35
      DO 1100 II=1,3
      DO 1100 JJ=1,3
      DO 1100 KK=1,3
      IF (COS(II,JJ,K).GE.C) GO TO 1100
      IIM=II
      JJM=JJ
      KKM=KK
      C=COS(II,JJ,KK)
 1100 CONTINUE
      IF (COS(2,2,2).EQ.C) GO TO 1500
      DELTSQ=DEL(TSQO)
      DELK=DEL(SKO)
      TSQO=TSQO+(IIM-2)*DELTSQ
      TSQD=TSQO-2.0*DELTSQ
      SNO=SNO+(JJM-2.0)
      IF (SNO.LT.3.0) SNO=3.0
      SNDL=SNO-2.0
      SKO=SKO+(KKM-2.0)*DELK
      IF (SKO.LE.0.005) SKO=0.005
      SKDL=SKO-2.0*DELK
      MIKE=MIKE+1
      IF (MIKE.GT.400) GO TO 1500
   25 FORMAT(I5)
   26 FORMAT(3E15.5)
   27 FORMAT('  PUNT',I5)
      IF (TSQD.GT..5) GO TO 1000
 9999 WRITE(6,27) MIKE
      C=10.**25
      SNM(I,J,K)=C
      SKM(I,J,K)=C
      TSQM(I,J,K)=C
      CMIN(I,J,K)=C
      GO TO 2000
 1500 SNM(I,J,K)=SNO
      SKM(I,J,K)=SKO
      TSQM(I,J,K)=TSQO
      CMIN(I,J,K)=C
      WRITE(6,25) MIKE
 2000 CONTINUE
      DO 2001 KDAX=1,3
 2001 WRITE(6,22)
C
C WRITE A PAGE OF OUTPUT
C
      WRITE(6,113) NP
      WRITE(6,14) (A1(I),I=1,3)
      WRITE(6,20)
      DO 2300 J=1,3
      WRITE(6,15) A2(J)
      DO  2200 K=1,3
      WRITE(6,16) A3(K),(CMIN(I,J,K),I=1,3)
```

```
      WRITE(6,17) (SNM(I,J,K),I=1,3)
      WRITE(6,18) (SKM(I,J,K),I=1,3)
      WRITE(6,19) (TSQM(I,J,K),I=1,3)
      IF (K.NE.3) WRITE(6,21)
 2200 CONTINUE
      WRITE(6,20)
 2300 CONTINUE
  500 CONTINUE
      GO TO 301
  501 CONTINUE
      END
```

APPENDIX C

APPROXIMATION FOR THE PROBABILITY INTEGRAL OF THE CENTRAL F DISTRIBUTION

Paulson (24) found an approximation for the probability integral of the central F distribution by developing a modified statistic x, a function of F, such that x is nearly normally distributed with zero mean and unit variance. That is

$$\int_{F_{\alpha},v_1,v_2}^{\infty} f(F,v_1,v_2)dF \doteq \int_{x}^{\infty} f(z)dz$$

where $f(F,v_1,v_2)$ is the frequency function of the central F distribution and f(z) is the frequency function of the normal distribution and x is given by

$$x = \frac{\left(1 - \frac{2}{9v_2}\right)F^{1/3} - \left(1 - \frac{2}{9v_1}\right)}{\left(\frac{2}{9v_2} F^{2/3} + \frac{2}{9v_1}\right)^{1/2}}$$

This approximation is used to calculate q_0 in the second stage computations with the following substitution

$$F = \frac{v_2 T^2}{v_1 (N-1)}$$

where v_1 and v_2 are the degrees of freedom.

To develop his approximation, Paulson (24) considered the F distribution as a ratio of two central chi-square distributions. That is,

$$F = \frac{\chi_1^2/v_1}{\chi_2^2/v_2}$$

with degrees of freedom v_1 and v_2, respectively. Wilson and Hilferty (27) showed that $(\chi^2/v)^{1/3}$ is nearly normally distributed with mean $(1-2/9v)$ and variance $2/9v$. From this, Paulson (24) obtained an approximation to the central F distribution by regarding $F^{1/3}$ as the ratio of two normally distributed variates. To accomplish this, Paulson (24) made use of work done by Fieller (8) who shows that if two normally and independently distributed variates x and y have means m_x and m_y and standard deviations, respectively, then

$$R = \frac{\left(\frac{y}{x}\right)m_x - m_y}{\left[\left(\frac{y}{x}\right)^2 \sigma_x^2 + \sigma_y^2\right]^{1/2}}$$

is nearly normally distributed with zero mean and unit variance. With the following substitutions

$$F^{1/3} = \frac{y}{x}$$

$$m_x = 1 - \frac{2}{9v_2}$$

$$m_y = 1 - \frac{2}{9v_1}$$

$$\sigma_x^2 = \frac{2}{9v_2}$$

$$\sigma_y^2 = \frac{2}{9v_1}$$

Paulson (24) found that

$$x = \frac{\left(1 - \frac{2}{9v_2}\right)F^{1/3} - \left(1 - \frac{2}{9v_1}\right)}{\left[\left(\frac{2}{9v_2}\right)F^{2/3} + \frac{2}{9v_1}\right]^{1/2}}$$

is nearly normally distributed with zero mean and unit variance.

An approximation for the probability integral of the noncentral F distribution was developed as an extension of Paulson's work. The noncentral F given by

$$F' = \frac{\chi_1^{2'}/v_1}{\chi_2^2/v_2}$$

is the ratio of a noncentral chi-square to a central chi-square distribution with v_1 and v_2 degrees of freedom, respectively. From Wilson and Hilferty (27) we know that (χ_2^2/v_2) is nearly normally distributed with mean $1 - \frac{2}{9v_2}$ and variance $\frac{2}{9v_2}$ Abdel-Aty (1) showed that $[\chi_1^{2'}/(v_1+\lambda)]$ is nearly normally distributed with mean

$$1 - \frac{2(v_1+2\lambda)}{9(v_1+\lambda)^2}$$

and variance

$$\frac{2(v_1+2\lambda)}{9(v_1+\lambda)^2}$$

where $\chi^{2'}$ is the noncentral chi-square with degrees of freedom v_1 and noncentrality parameter λ.

Therefore, if we consider the ratio

$$\frac{\chi_1^2/v_1}{\chi_2^2/v_2} = F'^{1/3} = \left(\frac{v_1 F'}{v_1}\right)^{1/3}$$

then the ratio

$$\left[\frac{\chi_1^{2'}/(v_1+\lambda)}{\chi_2^2/v_2}\right]^{1/3} = \left(\frac{v_1 F}{v_1+\lambda}\right)^{1/3}$$

To use the relationship developed by Feiller (8), the following substitutions are made:

$$\left(\frac{v_1 F}{v_1+\lambda}\right)^{1/3} = \left[\frac{\chi_1^{2'}/(v_1+\lambda)}{\chi_2^2/v_2}\right]^{1/3}$$

$$m_x = 1 - \frac{2}{9v_2}$$

$$\sigma_x^2 = \frac{2}{9v_2}$$

$$\sigma_y^2 = \frac{2(v_1+2\lambda)}{(v_1+\lambda)^2}$$

Then the quantity

$$x = \frac{\left(1 - \frac{2}{9v_2}\right)\left(\frac{v_1 F}{v_1+\lambda}\right)^{1/3} - \left[1 - \frac{2(v_1+2\lambda)}{9(v_1+\lambda)^2}\right]}{\left[\frac{2}{9v_2}\left(\frac{v_1 F}{v_1+\lambda}\right)^{2/3} + \frac{2(v_1+2\lambda)}{9(v_1+\lambda)^2}\right]^{1/2}}$$

is nearly normally distributed with zero mean and unit variance. Note that the expression above reduced to that of Paulson's when $\lambda = 0$.

APPENDIX D

PRELIMINARY ESTIMATES OF THE OPTIMAL VALUES OF THE TEST PARAMETERS

PLELIMINARY ESTIMATES FOR PROBLEM 1

COST A2	COST A3	PARAMETER	COST A1 .0001000	.0010000	.0100000
0000100					
	.001000	E*(C)	.02098	.03993	.11608
		N	7.0	9.0	9.0
		K	.03	.07	.30
		TSQR	9.07	7.45	7.45
	.010000	E*(C)	.03379	.05586	.12957
		N	9.0	9.0	9.0
		K	.03	.05	.30
		TSQR	21.82	21.82	7.45
	.100000	E*(C)	.13203	.16150	.25496
		N	9.0	9.0	9.0
		K	.02	.07	.10
		TSQR	49.57	21.82	21.82
0001000					
	.001000	E*(C)	.03583	.04931	.11861
		N	7.0	7.0	7.0
		K	.05	.07	.30
		TSQR	9.07	9.07	4.45
	.010000	E*(C)	.05406	.06847	.13227
		N	9.0	9.0	9.0
		K	.05	.07	.30
		TSQR	21.82	21.82	7.45
	.100000	E*(C)	.15962	.17306	.26158
		N	7.0	9.0	9.0
		K	.03	.10	.30
		TSQR	89.09	21.82	21.82
0010000					
	.001000	E*(C)	.09100	.10000	.13961
		N	5.0	5.0	7.0
		K	.10	.10	.30
		TSQR	6.08	6.08	4.45
	.010000	E*(C)	.12240	.12540	.15540
		N	7.0	7.0	7.0
		K	.30	.30	.30
		TSQR	9.07	9.07	9.07
	.100000	E*(C)	.24506	.25406	.28858
		N	9.0	9.0	9.0
		K	.10	.10	.30
		TSQR	21.82	21.82	21.82

PLELIMINARY ESTIMATES FOR PROBLEM 2

COST A2	COST A3	PARAMETER	COST A1 .0001000	.0010000	.0100000
0000100					
	.001000	E*(C)	.02069	.04021	.11332
		N	9.0	9.0	9.0
		K	.03	.07	.30
		TSQR	21.82	21.82	21.82
	.010000	E*(C)	.03283	.05082	.12411
		N	9.0	9.0	9.0
		K	.03	.07	.30
		TSQR	21.82	21.82	21.82
	.100000	E*(C)	.12898	.15251	.23207
		N	9.0	9.0	9.0
		K	.03	.05	.30
		TSQR	49.57	49.57	21.82
0001000					
	.001000	E*(C)	.03630	.04950	.11549
		N	5.0	5.0	5.0
		K	.05	.07	.30
		TSQR	6.08	6.08	6.08
	.010000	E*(C)	.04953	.06164	.12681
		N	9.0	9.0	9.0
		K	.07	.10	.30
		TSQR	21.82	21.82	21.82
	.100000	E*(C)	.15071	.16529	.23477
		N	9.0	9.0	9.0
		K	.05	.10	.30
		TSQR	49.57	21.82	21.82
0010000					
	.001000	E*(C)	.08567	.09467	.13049
		N	5.0	5.0	5.0
		K	.10	.10	.30
		TSQR	6.08	6.08	6.08
	.010000	E*(C)	.11309	.11848	.14848
		N	5.0	5.0	5.0
		K	.10	.30	.30
		TSQR	14.57	6.08	6.08
	.100000	E*(C)	.22877	.23177	.26177
		N	9.0	9.0	9.0
		K	.30	.30	.30
		TSQR	21.82	21.82	21.82

PLELIMINARY ESTIMATES FOR PROBLEM 3

COST A2	COST A3	PARAMETER	COST A1 .0001000	.0010000	.0100000
0000100					
	.001000	E*(C)	.02447	.04485	.12062
		N	9.0	9.0	7.0
		K	.03	.07	.30
		TSQR	21.82	21.82	9.07
	.010000	E*(C)	.03660	.05545	.13337
		N	9.0	9.0	9.0
		K	.03	.07	.30
		TSQR	21.82	21.82	21.82
	.100000	E*(C)	.13580	.16143	.24083
		N	9.0	9.0	9.0
		K	.02	.07	.30
		TSQR	49.57	21.82	21.82
0001000					
	.001000	E*(C)	.04035	.05392	.12272
		N	5.0	5.0	7.0
		K	.05	.07	.30
		TSQR	6.08	6.08	9.07
	.010000	E*(C)	.05416	.06689	.13607
		N	9.0	9.0	9.0
		K	.07	.10	.30
		TSQR	21.82	21.82	21.82
	.100000	E*(C)	.16014	.17040	.24353
		N	9.0	9.0	9.0
		K	.07	.10	.30
		TSQR	21.82	21.82	21.82
0010000					
	.001000	E*(C)	.09063	.09963	.13888
		N	5.0	5.0	5.0
		K	.10	.10	.30
		TSQR	6.08	6.08	6.08
	.010000	E*(C)	.12257	.12682	.15682
		N	5.0	5.0	5.0
		K	.10	.30	.30
		TSQR	6.08	6.08	6.08
	.100000	E*(C)	.23753	.24053	.27053
		N	9.0	9.0	9.0
		K	.30	.30	.30
		TSQR	21.82	21.82	21.82

PLELIMINARY ESTIMATES FOR PROBLEM 4

COST A2	COST A3	PARAMETER	COST A1 .0001000	COST A1 .0010000	COST A1 .0100000
0000100					
	.001000	E*(C)	.02983	.05046	.12722
		N	9.0	7.0	9.0
		K	.03	.07	.30
		TSQR	21.82	9.07	7.45
	.010000	E*(C)	.04195	.06176	.14072
		N	9.0	9.0	9.0
		K	.03	.07	.30
		TSQR	21.82	21.82	7.45
	.100000	E*(C)	.14168	.16751	.25139
		N	9.0	9.0	9.0
		K	.02	.07	.30
		TSQR	49.57	21.82	21.82
0001000					
	.001000	E*(C)	.04579	.05946	.12940
		N	5.0	7.0	7.0
		K	.05	.07	.30
		TSQR	6.08	9.07	9.07
	.010000	E*(C)	.06047	.07333	.14289
		N	9.0	9.0	7.0
		K	.07	.07	.30
		TSQR	21.82	21.82	9.07
	.100000	E*(C)	.16622	.17710	.25409
		N	9.0	9.0	9.0
		K	.07	.10	.30
		TSQR	21.82	21.82	21.82
0010000					
	.001000	E*(C)	.09690	.10590	.14830
		N	5.0	5.0	5.0
		K	.10	.10	.30
		TSQR	6.08	6.08	6.08
	.010000	E*(C)	.12878	.13389	.16389
		N	5.0	7.0	7.0
		K	.10	.30	.30
		TSQR	6.08	9.07	9.07
	.100000	E*(C)	.24809	.25109	.28109
		N	9.0	9.0	9.0
		K	.30	.30	.30
		TSQR	21.82	21.82	21.82

APPENDIX E

OPTIMAL ESTIMATES OF THE OPTIMAL VALUES OF THE TEST PARAMETERS

OPTIMAL ESTIMATES FOR PROBLEM 1

COST A2	COST A3	PARAMETER	COST A1 .0001000	COST A1 .0010000	COST A1 .0100000
0000100					
	.001000	E*(C)	.01889	.03882	.10811
		N	10.0	10.0	11.0
		K	.03	.06	.20
		TSQR	17.68	14.51	15.96
	.010000	E*(C)	.02908	.04860	.11821
		N	13.0	14.0	14.0
		K	.03	.07	.20
		TSQR	29.04	28.75	25.70
	.100000	E*(C)	.12133	.14189	.21742
		N	16.0	16.0	17.0
		K	.03	.06	.18
		TSQR	43.73	38.66	38.66
0001000					
	.001000	E*(C)	.03562	.04873	.11198
		N	7.0	7.0	6.0
		K	.05	.08	.22
		TSQR	8.16	8.16	4.89
	.010000	E*(C)	.05016	.06202	.12363
		N	10.0	10.0	10.0
		K	.07	.08	.22
		TSQR	17.67	17.67	14.51
	.100000	E*(C)	.15971	.15938	.22459
		N	8.0	13.0	14.0
		K	.04	.09	.20
		TSQR	72.17	29.04	28.75
0010000					
	.001000	E*(C)	.06956	.07661	.12863
		N	3.0	3.0	4.0
		K	.12	.13	.24
		TSQR	.82	.82	1.72
	.010000	E*(C)	.10815	.11389	.15348
		N	6.0	7.0	7.0
		K	.18	.20	.27
		TSQR	6.55	8.16	8.16
	.100000	E*(C)	.22473	.22940	.26867
		N	10.0	10.0	10.0
		K	.19	.19	.27
		TSQR	19.44	19.44	17.67

OPTIMAL ESTIMATES FOR PROBLEM 2

COST A2	COST A3	PARAMETER	COST A1 .0001000	COST A1 .0010000	COST A1 .0100000
0000100					
	.001000	E*(C)	.02051	.04013	.10728
		N	8.0	9.0	9.0
		K	.03	.06	.22
		TSQR	21.82	24.00	21.60
	.010000	E*(C)	.03050	.04981	.11733
		N	11.0	11.0	11.0
		K	.03	.06	.20
		TSQR	38.66	35.14	31.95
	.100000	E*(C)	.12255	.14305	.21682
		N	13.0	13.0	14.0
		K	.03	.06	.18
		TSQR	58.79	53.45	51.45
0001000					
	.001000	E*(C)	.03496	.04842	.11019
		N	5.0	5.0	5.0
		K	.05	.08	.22
		TSQR	8.90	8.09	6.69
	.010000	E*(C)	.04863	.06092	.12142
		N	8.0	8.0	9.0
		K	.06	.09	.22
		TSQR	24.00	21.82	24.00
	.100000	E*(C)	.14533	.15756	.22241
		N	11.0	11.0	11.0
		K	.06	.08	.20
		TSQR	44.17	42.52	38.27
0010000					
	.001000	E*(C)	.06895	.08078	.12618
		N	3.0	4.0	4.0
		K	.12	.15	.24
		TSQR	2.12	4.43	3.99
	.010000	E*(C)	.10061	.10566	.14643
		N	6.0	5.0	5.0
		K	.16	.18	.27
		TSQR	13.11	8.90	8.09
	.100000	E*(C)	.21310	.21818	.25920
		N	8.0	8.0	8.0
		K	.18	.18	.27
		TSQR	24.00	24.00	24.00

OPTIMAL ESTIMATES FOR PROBLEM 3

COST A2	COST A3	PARAMETER	COST A1 .0001000	COST A1 .0010000	COST A1 .0100000
0000100					
	.001000	E*(C)	.02433	.04431	.11310
		N	9.0	10.0	10.0
		K	.03	.06	.20
		TSQR	19.64	21.82	19.45
	.010000	E*(C)	.03442	.05401	.12319
		N	12.0	12.0	13.0
		K	.03	.06	.20
		TSQR	35.14	31.95	31.63
	.100000	E*(C)	.12653	.14728	.22245
		N	14.0	15.0	15.0
		K	.03	.06	.18
		TSQR	49.08	46.77	42.52
0001000					
	.001000	E*(C)	.04011	.05358	.11653
		N	5.0	5.0	7.0
		K	.05	.08	.22
		TSQR	6.69	6.08	9.98
	.010000	E*(C)	.05398	.06623	.12795
		N	9.0	9.0	9.0
		K	.06	.09	.22
		TSQR	21.82	19.64	17.67
	.100000	E*(C)	.15101	.16316	.22878
		N	11.0	12.0	12.0
		K	.06	.08	.20
		TSQR	35.14	35.14	31.95
0010000					
	.001000	E*(C)	.07434	.08659	.13620
		N	3.0	4.0	5.0
		K	.12	.15	.24
		TSQR	1.53	3.23	4.92
	.010000	E*(C)	.10878	.11506	.15592
		N	6.0	5.0	5.0
		K	.16	.18	.27
		TSQR	9.79	6.69	6.08
	.100000	E*(C)	.22335	.22837	.26904
		N	9.0	9.0	9.0
		K	.18	.20	.27
		TSQR	21.82	21.82	21.82

OPTIMAL ESTIMATES FOR PROBLEM 4

COST A2	COST A3	PARAMETER	COST A1 .0001000	.0010000	.0100000
0000100					
	.001000	E*(C)	.02932	.04942	.11905
		N	10.0	10.0	10.0
		K	.03	.06	.20
		TSQR	21.82	19.45	17.38
	.010000	E*(C)	.03941	.05916	.12914
		N	12.0	13.0	13.0
		K	.03	.06	.20
		TSQR	31.95	31.95	28.27
	.100000	E*(C)	.13159	.15241	.22830
		N	15.0	15.0	16.0
		K	.03	.06	.18
		TSQR	48.59	42.52	42.52
0001000					
	.001000	E*(C)	.04520	.05874	.12261
		N	6.0	6.0	7.0
		K	.05	.08	.20
		TSQR	8.09	7.35	8.98
	.010000	E*(C)	.05966	.07191	.13409
		N	9.0	9.0	10.0
		K	.06	.08	.21
		TSQR	19.64	17.67	19.45
	.100000	E*(C)	.15675	.16889	.23501
		N	12.0	12.0	13.0
		K	.06	.08	.20
		TSQR	35.14	31.95	31.95
0010000					
	.001000	E*(C)	.07999	.08711	.13935
		N	3.0	3.0	4.0
		K	.12	.13	.24
		TSQR	1.25	1.13	2.36
	.010000	E*(C)	.11596	.12115	.16224
		N	6.0	6.0	6.0
		K	.16	.18	.27
		TSQR	8.09	8.08	7.35
	.100000	E*(C)	.23119	.23621	.27703
		N	9.0	9.0	9.0
		K	.18	.20	.27
		TSQR	19.64	19.64	19.64

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